

# Physics and the choice of regulators in functional renormalisation group flows

Jan M. Pawłowski,<sup>1,2</sup> Michael M. Scherer,<sup>1</sup> Richard Schmidt,<sup>3,4</sup> and Sebastian J. Wetzel<sup>1</sup>

<sup>1</sup>*Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany*

<sup>2</sup>*ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum für*

*Schwerionenforschung mbH, Planckstr. 1, D-64291 Darmstadt, Germany*

<sup>3</sup>*ITAMP, Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, MA 02138, USA*

<sup>4</sup>*Physics Department, Harvard University, 17 Oxford Street, Cambridge, MA 02138, USA*

The Renormalisation Group is a versatile tool for the study of many systems where scale-dependent behaviour is important. Its functional formulation can be cast into the form of an exact flow equation for the scale-dependent effective action in the presence of an infrared regularisation. The functional RG flow for the scale-dependent effective action depends explicitly on the choice of regulator, while the physics does not. In this work, we systematically investigate three key aspects of how the regulator choice affects RG flows: (i) We study flow trajectories along closed loops in the space of action functionals varying both, the regulator scale and shape function. Such a flow does not vanish in the presence of truncations. Based on a definition of the length of an RG trajectory, we suggest a practical procedure for devising optimised regularisation schemes within a truncation. (ii) In systems with various field variables, a choice of relative cutoff scales is required. At the example of relativistic bosonic two-field models, we study the impact of this choice as well as its truncation dependence. We show that a crossover between different universality classes can be induced and conclude that the relative cutoff scale has to be chosen carefully for a reliable description of a physical system. (iii) Non-relativistic continuum models of coupled fermionic and bosonic fields exhibit also dependencies on relative cutoff scales and regulator shapes. At the example of the Fermi polaron problem in three spatial dimensions, we illustrate such dependencies and show how they can be interpreted in physical terms.

## I. INTRODUCTION

In the past twenty years, the functional renormalisation group (FRG) approach [1] has been established as a versatile method allowing to describe many aspects of different physical systems in the framework of quantum field theory and statistical physics. Applications range from quantum dots and wires, statistical models, condensed matter systems in solid state physics and cold atoms over quantum chromodynamics to the standard model of particle physics and even quantum gravity. For reviews on the various aspects of the functional RG see [2–19].

The functional renormalisation group approach can be set-up in terms of an exact flow equation for the effective action of the given theory or model [1]. The choice of the initial condition at some large ultraviolet cutoff scale, typically a high momentum or energy scale, together with that of the regulator function determines both, the physics situation under investigation as well as the regularisation scheme. The functional RG flow for the scale-dependent effective action depends explicitly on the choice of regulator, while the physics does not. The latter is extracted from the full quantum effective action at vanishing cutoff scale implying a vanishing regulator. Hence, at this point no dependence on the choice of regulator is left, only the implicit choice of the regularisation scheme remains.

Typically, for the solution of the functional flow equation for the effective action one has to resort to approximations to the effective action as well as to the flow. Such a truncation of the full flow usually destroys the regulator independence of the full quantum effective action at

vanishing cutoff. Therefore, devising suitable expansion schemes and regulators is essential for reliable results. Moreover, the related considerations also allow for a discussion of the systematic error within a given truncation scheme. This has been examined in detail for the computation of critical exponents in models with a single scalar field in three dimensions within the lowest order of the local potential approximation (LPA), [7, 20–28]: an optimisation procedure, [7, 22] suggests a particular regulator choice – the flat regulator – which is also shown to yield the best results for the critical exponents.

The optimisation framework in [7] has been extended to general expansion schemes in a functional optimisation procedure including fully momentum-dependent approximation schemes. An application to momentum-dependent correlation functions in Yang-Mills theory can be found in [29].

Still, for more elaborate truncations, in particular higher orders of the derivative expansion in the LPA, including, e.g., momentum dependencies or higher-order derivative terms, little has been done when it comes to a practical implementation of optimisation criteria. Also, more complex physical models with different symmetries such as, e.g., non-relativistic systems, or models with several different fields, for example mixed boson-fermion systems, demand for a thorough study of their regulator dependence in order to extract the best physical results from a given truncation.

In this paper we study the impact of different regulator choices on truncated functional renormalisation group flows in various models and further develop the functional optimisation procedure set-up in [7]. In its present form

it allows for a practical and simple comparison of the quality of different regulators and for the construction of an optimised one.

The rest of the paper is organised as follows: In Sec. II we shortly introduce the FRG approach and explain how the choice of a specific regulator influences a truncated FRG flow. This is captured in terms of an integrability condition for closed loops in theory space upon a change of the regulator and RG scale, cf. Sec. IID. In Sec. III we then devise a road towards a practical optimisation procedure. We discuss the length of an RG trajectory which has to be minimal for an optimised regulator, cf. Sec. IIIB. This procedure is then applied to a simply scalar model (Sec. IIIC). A more heuristic approach to models with various degrees of freedom such as two-scalar models and non-relativistic boson-fermion systems, is presented in Secs. IV and V, respectively. To this end, we introduce a shift between the regulator scales of the different fields and show how this affects the results allowing for a change of the underlying physics upon varying the regulator. This, again, clearly demands for carefully choosing a regularisation scheme which could be performed by a optimisation procedure as suggested in this work.

## II. FUNCTIONAL RG FLOWS

The functional renormalisation group is based on the Wilsonian idea of integrating out degrees of freedom. In the continuum, this idea can be implemented by suppressing the fluctuations in the theory below an infrared cutoff scale  $k$ . An infinitesimal change of  $k$  is then described in terms of a differential equation for the generating functional of the theory at hand – Wetterich’s flow equation [1]. The infrared suppression can be achieved by adding a momentum-dependent mass term to the classical action,

$$S[\varphi] \rightarrow S[\varphi] + \frac{1}{2} \int_p \varphi(p) R_k(p) \varphi(-p), \quad (1)$$

with  $\int_p = \int \frac{d^d p}{(2\pi)^d}$ . The regulator  $R_k(p)$  tends towards a mass for low momenta and vanishes sufficiently fast in the ultraviolet, see (3) and (4).

With the cutoff scale dependent action (1) also the one-particle-irreducible (1PI) effective action or free energy,  $\Gamma_k[\phi]$ , acquires a scale dependence. The  $n$ th field derivatives of the effective action,  $\Gamma_k^{(n)}[\phi]$ , are the 1PI parts of the  $n$ -point correlation functions in a general background  $\phi = \langle \varphi \rangle$ . The flow of  $\Gamma_k$  is given by

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} G_k[\phi] \partial_t R_k \text{ with } G_k[\phi] = \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}, \quad (2)$$

where we have introduced the renormalisation time  $t = \ln k/\Lambda$ . Here,  $\Lambda$  is some reference scale, usually the ultraviolet scale, where the flow is initiated. The trace sums over all occurring indices, including the loop integration

over momenta. The regulator is conveniently written as

$$R_k(p) = p^2 r(p^2/k^2), \quad (3)$$

with the dimensionless shape function  $r(y)$  that only depends on the dimensionless ratio  $y = p^2/k^2$ . The regulator functions fulfill the infrared and ultraviolet conditions

$$\lim_{y \rightarrow 0} y r(y) > 0, \quad \lim_{y \rightarrow \infty, \epsilon > 0} y^{d/2+\epsilon} r(y) = 0. \quad (4)$$

The first limit in (4) implements the infrared suppression of low momentum modes as the propagator  $G_k$  acquires an additional infrared mass due to  $R_k$ . The second limit guarantees that the ultraviolet is unchanged. The regulator  $R_k(p)$  has to decay with higher powers as  $p^d$  in  $d$  dimensions in order to have a well-defined flow equation without the need of an ultraviolet renormalisation. With (4) the flow equation (2) is ultraviolet finite due to the sufficiently fast decay of the regulator in the ultraviolet. Here, we presented the relativistic case for simplicity. The arguments can be extended to the non-relativistic case, e.g. [30]. We discuss one specific example for such a non-relativistic system in Sec. V.

### A. Ultraviolet Limit and Regulator Dependence

In the limit  $k \rightarrow \infty$ , the cutoff term in (1) suppresses all momentum fluctuations. To discuss this limit, we consider the RG running of the scale- $k$ -dependent couplings  $g_n(k)$  parametrising the theory in terms of a suitable basis of field monomials, i.e.,  $\Gamma_k = \sum_n g_n(k) \mathcal{O}_n(\partial, \phi)$ . We classify the  $g_n(k)$  according to their UV scaling dimension  $d_n$  that follows from the running of the couplings towards the ultraviolet (UV) with the flow equation (2). The UV scaling dimension  $d_n$  is the full quantum dimension, i.e., canonical plus anomalous dimension,

$$g_n(k) \sim k^{d_n}. \quad (5)$$

Terms in the effective action  $\Gamma_k$  whose couplings  $g_n(k)$  have semi-positive UV scaling dimension,  $d_n \geq 0$ , dominate the UV behaviour. In turn, terms with couplings  $g_n(k)$  with  $d_n < 0$  are sub-leading or suppressed.

Let us elucidate this at the example of the relativistic  $\varphi^4$  field theory in  $d = 3$  dimensions. This theory is super-renormalisable and the only parameter with a positive UV scaling dimension is the mass parameter  $m_k^2 = \Gamma_k^{(2)}(p = 0, \phi = 0)$ . The flow of the mass is derived from (2) with a second order field derivative evaluated at vanishing fields and momenta, to wit

$$\partial_t m_k^2 = -\frac{1}{2} \int_q G_k(q) \partial_t R_k(q) G_k(q) \Gamma_k^{(4)}(q, q, 0, 0), \quad (6)$$

where we have used that the three-point function vanishes due to the symmetry of the theory under  $\phi \rightarrow -\phi$ , i.e.,  $\Gamma_k^{(3)}[\phi = 0] = 0$ . In the UV limit the flow (6) simpli-

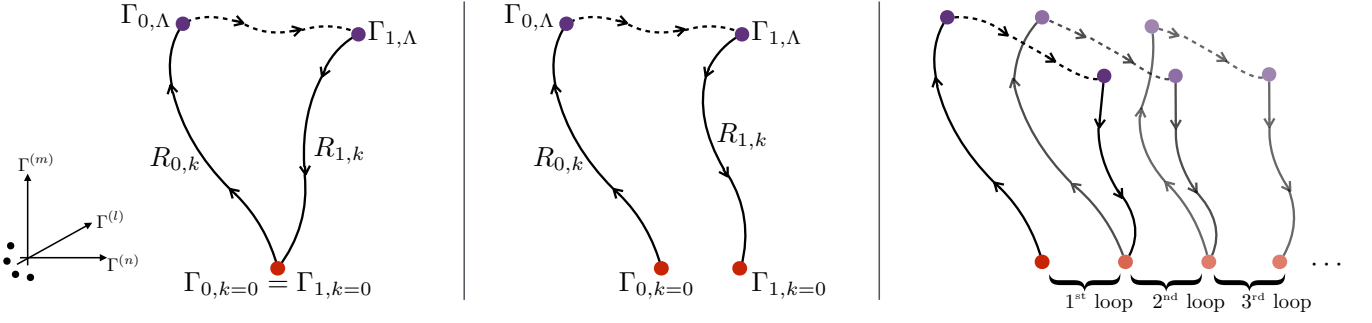


FIG. 1: Left panel: Pictorial representation of the integrability condition (12) in the theory space of action functionals. By means of (11), we can map the two actions at the initial scale  $\Lambda$  onto each other. Integrating out momentum fluctuations from  $k = \Lambda$  toward  $k = 0$  then yields the full quantum effective action for both regulators, i.e.,  $\Gamma_{0,k=0} = \Gamma_{1,k=0}$ . Middle panel: Approximations lead to a violation of the integrability condition, cf. (15). Right panel: Pictorial representation of several successively performed loops according to (15) within an approximation for the full propagator  $G_k$ . This sketch exhibits how the result for the full effective action  $\Gamma_{k=0}$  moves further and further away from its initial position in theory space.

fies considerably. The four-point function tends towards a local scale-independent vertex, i.e.,  $\Gamma_{k \rightarrow \infty}^{(4)} \rightarrow \lambda_{UV}$ , up to momentum conservation. For  $k \rightarrow \infty$ , the propagators in (6) are simply given by

$$G_k(q) = \frac{1}{q^2[1 + r(q^2/k^2)] + m_k^2}. \quad (7)$$

Here, we have also used that the wave function renormalisation  $Z_k \rightarrow 1 + O(1/k)$  in the limit  $k \rightarrow \infty$ . This can be proven analogously to the following determination of the asymptotic scaling of the mass. For the mass we are hence led to the asymptotic UV flow

$$\partial_t m_k^2 = -k \lambda_{UV} \int_{\bar{q}} \frac{\bar{q}^4 \partial_{\bar{q}^2} r(\bar{q}^2)}{(\bar{q}^2[1 + r(\bar{q}^2)] + \bar{m}_k^2)^2}, \quad (8)$$

where quantities denoted with a bar are scaled with appropriated powers of  $k$  in order to make them dimensionless, i.e.,  $\bar{q}^2 = q^2/k^2$  and  $\bar{m}_k^2 = m_k^2/k^2$ . The flow (8) is further simplified if we reduce it to the leading UV scaling. To that end, we notice that the flow scales with  $k$  for  $\bar{m}_k = 0$ . Hence, for  $k \rightarrow \infty$  we have  $m_k^2 \propto \lambda_{UV} k$  and  $\bar{m}_k^2 \propto \lambda_{UV}/k \rightarrow 0$ . Accordingly, we have

$$m_k^2 = \mu(r) \lambda_{UV} k + O(k^0),$$

with the dimensionless factor

$$\mu(r) = - \int_{\bar{q}} \frac{\partial_{\bar{q}^2} r(\bar{q}^2)}{[1 + r(\bar{q}^2)]^2}. \quad (9)$$

We conclude that the mass parameter  $m_k^2$  diverges linearly with  $k$ . We also note that the constant  $\mu(r)$  is non-universal and depends on the chosen regulator  $r$ . Additionally, the above simple example nicely reflects the regularisation and renormalisation scheme dependence in the present modern functional RG setting: UV divergences in standard perturbation theory are reflected in UV relevant terms such as  $\mu(r) \lambda_{UV} k$ , that diverge

for  $k \rightarrow \infty$ . The subtractions or renormalisation in perturbation theory are reflected in the consistent choice of the initial condition that makes the full effective action  $\Gamma_{k=0}$  independent of the initial scale  $k = \Lambda$ . Accordingly, the initial mass  $m_\Lambda^2$  has to satisfy the flow equation (6) which again leads to (9) for  $m_\Lambda^2$ . In other words, the  $\Lambda$ -dependence of the flow is annihilated by that of the initial conditions. This accounts for a BPHZ-type renormalisation, for detailed discussions see e.g. [7, 13, 31]. Consequently, a part of the standard renormalisation scheme dependence is carried by the regulator dependence of  $\mu(r)$ .

Finally, the physics is entirely carried by the finite part of the UV limit, that is the  $O(k^0)$  term in (9). Since this finite UV part has first to be mapped to  $k \rightarrow 0$  via the flow, it also carries a renormalisation scheme dependence. In summary, the latter is given by a combination of the shape dependence and the finite part of the initial condition. This simple distinction can be used to rewrite the effective action in terms of renormalised fields and parameters for obtaining a finite UV limit, see e.g. [7, 13].

## B. Initial Actions and Integrability Condition

The above discussion already highlights the regulator- or  $r$ -dependence of the flow and the scale-dependent effective action. However, despite this  $r$ -dependence of the flow, the final effective action  $\Gamma_{k=0}$  is unique up to RG transformations, see App. A for a discussion of this issue. This is illustrated in theory space in the left panel of Fig. 1. The initial effective actions at the UV scale  $\Lambda$  differ due to the different shape functions  $r$ . Nonetheless, we can map the initial effective actions onto each other by the following flow equation

$$\partial_s \Gamma_{s,\Lambda} = \frac{1}{2} \text{Tr} G_\Lambda[\phi] \partial_s R_{s,\Lambda}, \quad (10)$$

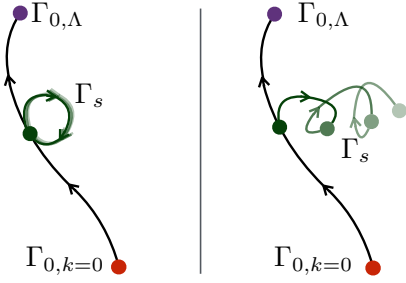


FIG. 2: Generalized one-parameter flows. Left panel: The exact flow equation provides the integrability condition, (15). Right panel: In general, the integrability condition is violated when approximations to  $G_k$  are employed, cf. (16).

with  $R_{s,k} = p^2 r_s$  and where we use a one parameter family of shape functions  $r_s(y)$  with  $r_0(y) = r^A(y)$  and  $r_1(y) = r^B(y)$  which analytically transforms  $r^A$  into  $r^B$ . Then, (10) is easily derived similarly to the cutoff flow (2). Consequently, the two initial actions are mapped onto each other by the  $s$ -integration of (10),

$$\Gamma_{1,\Lambda} = \Gamma_{0,\Lambda} + \frac{1}{2} \int_0^1 ds \text{Tr} G_\Lambda[\phi] \partial_s R_{s,\Lambda}, \quad (11)$$

and the full quantum effective actions agree trivially,  $\Gamma_{0,k=0} = \Gamma_{1,k=0}$ , see also left panel of Fig. 1. This statement can be reformulated as an integrability condition

$$\int_0^1 \partial_s \Gamma_{s,\Lambda} + \int_\Lambda^0 \frac{dk}{k} \partial_t \Gamma_{1,k} + \int_0^\Lambda \frac{dk}{k} \partial_t \Gamma_{0,k} = 0, \quad (12)$$

which defines a closed loop in theory space.

### C. Integrability Condition and Approximations

In general, the integrability condition (12) is violated in approximations to the full effective action, and the flow ceases to describe a total derivative with respect to  $t$  and  $s$ . The relation between an approximation of the effective action or rather of the propagator in the flow equation and the derivative operators  $\partial_t$  and  $\partial_s$  is more clearly seen in the general flow equation for composite operators  $I_k$ . This set includes general  $n$ -point correlation functions  $\langle \phi(x_1) \cdots \phi(x_n) \rangle$  with connected and disconnected parts [7]. A further relevant example is  $\delta \Gamma_k / \delta \phi$ . The flow equation for the  $I_k$  reads

$$\partial_t I_k[\phi] = \left( -\frac{1}{2} \text{Tr} G_k[\phi] \partial_t R_k G_k[\phi] \frac{\delta^2}{\delta \phi^2} \right) I_k. \quad (13)$$

For  $\delta \Gamma_k / \delta \phi$  one easily sees that the  $\phi$ -derivative of the flow (2) gives (13) with  $I_k = \Gamma_k^{(1)}$ . Note, however, that the effective action  $\Gamma_k$  does not satisfy (13). Another simple test is given by the full two-point function  $G_k(p, q) +$

$\phi(p)\phi(q)$ . Importantly, as the set of composite operators that satisfy (13) includes all correlation functions it is complete. We conclude that the total  $t$ -derivative has the representation

$$\partial_t = \left( -\frac{1}{2} \text{Tr} G_k[\phi] \partial_t R_k G_k[\phi] \frac{\delta^2}{\delta \phi^2} \right), \quad (14)$$

on the – complete – set of composite operators  $\{I_k\}$ . Eq. (14) makes explicit the consequences of approximations to the effective action for the total  $t$ -derivative: they imply approximations for the full propagators  $G_k$  and hence approximations to the representation (14) of the total  $t$ -derivative  $\partial_t$ . Consequently, an integration along a closed loop in regulator space does not necessarily vanish within an approximation to the effective action. Note that the notation  $\partial_t$  for the total  $t$ -derivative is common in the FRG community and we keep it for the sake of comparability.

For our discussion of flows that change regulators as well as cutoff scales we extend the notation with the parameter  $s$  to general one-parameter flows in theory space. Such a flow includes changes of the cutoff scale  $k$  with  $k(s)$ , changes of the shape of the regulator  $r_s$  and reparametrisations of the theory, see App. B for a detailed discussion. Within this unified approach a closed loop such as the global one in (12) has the simple representation

$$\oint_{\mathcal{C}} d\Gamma[\phi, R] = \int_0^1 ds \frac{d}{ds} \Gamma_s[\phi, R_s] = 0, \quad (15)$$

Here,  $s$  parameterises the closed curve  $\mathcal{C}$ , and  $R_s$  describes a loop in regulator space with  $R_0 = R_1$ . In general, approximations to the effective action  $\Gamma_k$  lead to

$$\int_0^1 ds \frac{d}{ds} \Gamma_s[\phi, R_s] \neq 0, \quad (16)$$

for closed loops, see Fig. 2 for a pictorial representation. This also means that if such a loop is taken many periods eventually the result will be dominated completely by the errors introduced by the approximation of the representation to the total  $t$ -derivative. In particular, the global loop shown in the left panel of Fig. 1 does not close.

A violation (16) of the integrability condition (15) is a measure for the self-consistency of the approximation at hand. In the following we will use it in our quest for optimal regulators as well as a systematic error estimate. In App. C we discuss under which circumstances (15) is violated and when it is satisfied. In summary, the violation of the integrability condition measures the incompleteness, in terms of the full quantum theory, of fully non-perturbative resummation schemes. This allows for a systematic error estimate of a given approximation: Consider general closed loops in theory space initiated from a given regulator  $R_k^A$ . Then, we change the regulator at a fixed initial scale as in (10), and subsequently flow to vanishing cutoff scale. For sensible regulator choices,

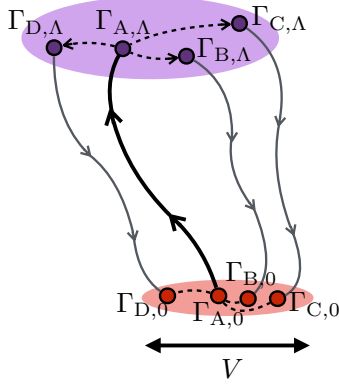


FIG. 3: A change of regulator at the UV scale  $\Lambda$ , Eq. (11), maps the action  $\Gamma_{A,\Lambda}$  onto one of the others,  $\Gamma_{B,\Lambda}, \Gamma_{C,\Lambda}, \Gamma_{D,\Lambda}$ , corresponding to different regulators. Then, integrating the RG flow in an approximation towards  $k = 0$  results in different approximations to the IR quantum effective action, i.e.,  $\Gamma_{i,0}$  with  $i \in \{A, B, C, D\}$ , cf. Eq. (15). The spreading of these results from a large class of general regulators can be used as an error estimate for an approximation.

the spreading of the results for  $\Gamma[\phi, R = 0]$  provides an error estimate, see Fig. 3 for a pictorial representation.

#### D. Scalar model

Generally, it is not possible to exactly solve the flow equation (2) for the flowing action  $\Gamma_k$ . Therefore, we have to devise suitable truncation schemes for the functional  $\Gamma_k$ . A simple scheme is given by the derivative expansion which assumes the smallness of momentum fluctuations. In this section, we will investigate a three-dimensional  $O(1)$  symmetric scalar model to explore the effects of regulator choices on functional RG results. Our ansatz is given by the local potential approximation (LPA)

$$\Gamma_k = \int_x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + U_k(\phi) \right], \quad U_k = \sum_i \frac{\bar{\lambda}_i}{i!} (\bar{\rho} - \bar{\kappa})^i, \quad (17)$$

with the real scalar field  $\phi$ , the scale-dependent effective potential  $U_k$  and the field invariant  $\bar{\rho} = \frac{1}{2} \phi^2$ .

The expansion parameters of the effective potential are scale dependent quantities  $\bar{\lambda}_n = \bar{\lambda}_n(k)$  and  $\bar{\kappa} = \bar{\kappa}(k)$ , however for brevity, we will not indicate this in the following. Further, we have set the wave function renormalisation to unity, dropping all non-trivial momentum-dependences. For calculations, we introduce the dimensionless effective potential and couplings  $u(\rho) = U_k k^{-d}$ ,  $\kappa = k^{2-d} \bar{\kappa}$ ,  $\rho = k^{2-d} \bar{\rho}$  and  $\lambda_i = \bar{\lambda}_i k^{-d+i(d-2)}$ . Then, we can write the flow equation for the effective potential as

$$\partial_t u = -du + (d-2)\rho u' + I(u' + 2\rho u''), \quad (18)$$

where primes denote derivatives with respect to  $\rho$  and we

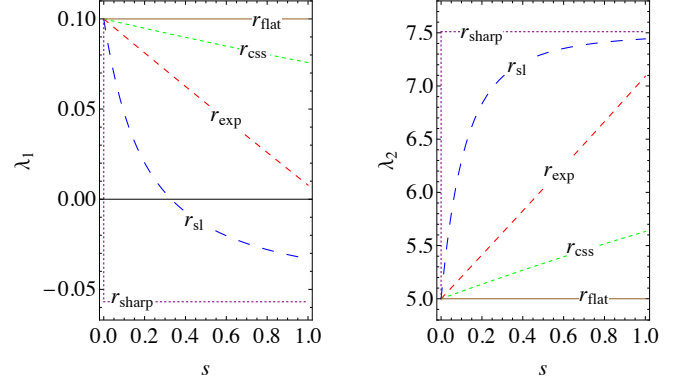


FIG. 4: Change of the initial conditions for different choices of interpolations from  $r_{\text{flat}}$  to  $r_i$ . The parameter-dependent regulator  $r_{\text{css}}$  was chosen with  $c = 0.5, b = 1, h = 0.5$ . The shape function  $r_{\text{sl}}$  with  $c = 10$ .

have defined the threshold function

$$I(w) = v_d \int_0^\infty y^{\frac{d}{2}+1} dy \frac{-2 \partial_y r(y)}{y(1+r(y)) + w}, \quad (19)$$

with  $y = p^2/k^2$  and  $v_d^{-1} = 2^{d+1} \pi^{\frac{d}{2}} \Gamma(d/2)$ . Using the series expansion of the effective potential, (17), also for its dimensionless version, we extract the flow equations for the individual couplings by projections, see App. E for explicit expressions.

##### 1. Switching Regulators at Fixed RG Scale

As discussed in Sect. II B a change in the regulator triggers a flow in the space of action functionals. In particular this means that switching from one regulator to another induces a change in the initial conditions as exhibited in (11). To visualise this change explicitly, we employ superpositions of two regulators at a fixed scale  $k$ , with  $s \in [0, 1]$ ,

$$r_s(y) = s r^A(y) + (1-s) r^B(y). \quad (20)$$

The flow equation with respect to the variable  $s$  is then,

$$\partial_s u = J(u' + 2\rho u''),$$

$$J(w) = v_d \int_0^\infty y^{\frac{d}{2}} dy \frac{\partial_s r_s(y)}{y(1+r_s(y)) + w}, \quad (21)$$

where  $\partial_s r_s(y) = r^A(y) - r^B(y)$ . More generally, we do not require a linear superposition as specified in (20), but we can switch regulators on an arbitrary smooth trajectory while keeping the scale  $k$  fixed. The change of initial conditions from switching between different regulators is then given by the solution of the flow equation (21).

In Fig. 4, we show this solution for a collection of representative regulator shape functions listed in Tab. I.

Regulator type	Representation	Limits
exponential	$r_{\text{exp}}(y) = (\exp(y) - 1)^{-1}$	-
flat (Litim)	$r_{\text{flat}}(y) = (\frac{1}{y} - 1)\theta(1 - y)$	-
step-like	$r_{\text{sl}}(y) = \frac{c}{y}\theta(1 - y)$	$\lim_{c \rightarrow \infty} r_{\text{sl}}(y) \rightarrow r_{\text{sharp}}(y)$
compactly supported smooth	$r_{\text{css}}(y) = \frac{c\theta(1 - hy^b)}{\exp(\frac{cy^b}{1 - hy^b}) - 1}$	(i) $\lim_{b \rightarrow \infty} r_{\text{css}}(y) \rightarrow r_{\text{sharp}}(y)$ , for $c > 0, h = 0$ (ii) $\lim_{c \rightarrow 0} r_{\text{css}}(y) \rightarrow r_{\text{flat}}(y)$ , for $b = 1, h = 1$ (iii) $r_{\text{css}}(y) = r_{\text{exp}}(y)$ , qfor $b = c = 1, h = 0$
exponential interpolating	$r_{\text{int}}(y) = \frac{(a - by)y^{n-1}}{\exp(y^n) - 1}$	(i) $\lim_{a, n \rightarrow \infty} r_{\text{int}}(y) \rightarrow r_{\text{sharp}}(y)$ , for $b = 0$ (ii) $\lim_{n \rightarrow \infty} r_{\text{int}}(y) \rightarrow r_{\text{flat}}(y)$ , for $a = 1, b = 1$ (iii) $r_{\text{int}}(y) = r_{\text{exp}}(y)$ , for $a = 1, b = 0, n = 1$

TABLE I: Classes of regulator shape functions  $r$ , and parameter choice that provide exponential, flat and sharp regulators in the respective class.

Here, we have integrated flow equations within an LPA to fourth order in the fields  $\phi^4$  in the symmetric regime, concentrating on the four-scalar coupling and the only relevant coupling, the mass parameter. In Fig. 4 we follow the regulator-dependence of the two dimensionless couplings  $\lambda_1 = m^2$  and  $\lambda_2$ , starting at  $s = 0$  with the flat regulator  $r_{\text{flat}}$  and the initial conditions  $\lambda_1^{(\text{in})} = 0.1$  and  $\lambda_2^{(\text{in})} = 5.0$ . Fig. 4 clearly exhibits the change in the initial conditions upon variations of the regulator shape function at fixed RG scale  $k$ . Interestingly, the largest difference in initial conditions starting from  $r_{\text{flat}}$  is given by switching to the sharp regulator  $r_{\text{sharp}}$ .

## 2. Loops in $k - R_k$ Space

In addition to the change of the regulator shape from  $R_k^A$  to  $R_k^B$  at a fixed RG scale, we now allow for a dependence of the RG scale  $k$  on the loop variable  $s$ , i.e.,  $k \rightarrow k(s)$ . Then, we can perform integrations along closed loops in theory space, cf. Fig. 2, and study the violation of the integrability condition, (15), explicitly. Such a combined change of regulator and RG scale can be incorporated in a linear interpolation between two scale dependent regulators

$$R_{s,k(s)} = a(s)R_{k(s)}^A + (1 - a(s))R_{k(s)}^B, \quad (22)$$

where  $a(s) \in [0, 1]$  parametrises the switching from one regulator function to another. In order to solve the flow equations along a loop in  $k - R_k$ -space we add to (21) the terms which include solving the flow equations in  $k$  direction,

$$\frac{d}{ds}u = J(u' + 2\rho u'') + \frac{\partial_s k(s)}{k(s)}(-du + (d-2)\rho u'). \quad (23)$$

The  $s$ -derivative of the regulator in the threshold function  $J(\omega)$  defined in (21) can be decomposed into two contributions

$$\frac{d}{ds}r_s(y) = \underbrace{\partial_s r_s(y)}_{\text{change of shape}} - 2 \underbrace{\frac{\partial_s k(s)}{k(s)}y \partial_y r_s(y)}_{\text{explicit change of scale}}, \quad (24)$$

where the first term keeps track of the change of the shape of the regulator function  $r_s(y)$ , while the second term tracks the change of the cutoff scale  $k(s)$ . Evidently this is just a convenient splitting as the change of  $k(s)$  can also be easily described by a change of  $r_s$ . This is seemingly a trivial remark but it hints at the fact that a change of the shape of the regulator may very well imply a change of the physical cutoff scale. This discussion will be detailed further in Section III. Explicitly, the involved derivatives are given by

$$\begin{aligned} \partial_s r_s(y) &= (r^A(y) - r^B(y))\partial_s a(s), \\ \partial_y r_s(y) &= a(s)\partial_y r^A(y) + (1 - a(s))\partial_y r^B(y). \end{aligned} \quad (25)$$

In the following, we again employ a linear superposition between two regulators, e.g.,  $r_{\text{exp}}$  and  $r_{\text{sharp}}$ , and solve the flow of the scalar model in LPA to order  $\phi^4$  along a closed loop. Each closed loop in  $k - R_k$  space along a rectangle contour then consists of four steps, see Fig. 5 for a representative contour:

1. The flow equation is solved from  $k_1$  to  $k_2$ .
2. Switch the regulator continuously from  $r^A$  to  $r^B$ .
3. Reverse the flow from  $k_2$  to  $k_1$ .
4. Switch back from regulator  $r^B$  to  $r^A$ .

The flow of the couplings  $\lambda_1$  and  $\lambda_2$  along one such closed loop is depicted in Fig. 6. For these calculations, we again

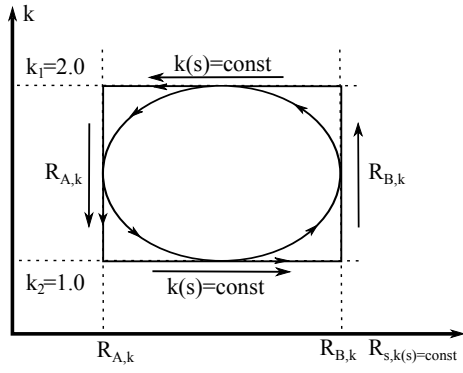


FIG. 5: Representative contours of a closed loop in  $k - R_k$  space including changes of regulator shape as well as RG scale. The rectangle contour separates the flow equations and switching the regulators. The ellipse contour however, allows for simultaneous changes in the scale and the regulator shape.

use the initial values  $\lambda_1^{(\text{in})} = 0.1$  and  $\lambda_2^{(\text{in})} = 5.0$ . We switch from  $r_{\text{flat}}$  to  $r_{\text{sharp}}$  (red dashed) or to  $r_{\text{exp}}$  (blue solid), respectively. The change from  $r_{\text{flat}}$  to  $r_{\text{exp}}$  is smooth as both regulators are finite for all momenta. In contrast, the change from  $r_{\text{flat}}$  to  $r_{\text{sharp}}$  shows a discontinuous peak in the flow of  $\lambda_{1,2}$ : the transition from the flat to the sharp regulator instantly lends an infinite infrared mass to the propagator for momenta lower than the cutoff scale of the sharp regulator. In either case the integration along one of our chosen closed-loop contours shows slight deviations from the initial values of  $\lambda_1$  and  $\lambda_2$ , see Fig. 6.

The deviations from the initial values add up when the procedure of integrating along a closed-loop contour in  $k - R_k$  space is repeated. This is shown in the upper panel of Fig. 7 for a consecutive integration along four of the closed loops as defined in Fig. 5. In fact, after these four closed loops the values of the coupling constants  $\lambda_{1,2}$  strongly deviate from their initial values. For comparison, in Fig. 7, we also show an integration along an alternative closed-loop, defined by an ellipse contour, cf. Fig. 5. This integration can be performed in a completely analytical way for a transition from  $r_{\text{flat}}$  to  $r_{\text{sharp}}$  as shown in App. F.

Our study clearly demonstrates the violation of the integrability condition, (15), for truncated renormalisation group flows. The severeness of this violation depends on the chosen regulators, cf. Fig. 6 and indicates the necessity of an educated choice of the regularisation scheme in renormalisation group investigations to establish and improve the reliability of physical results. The following section is dedicated to devising such an educated choice in terms of an optimisation procedure.

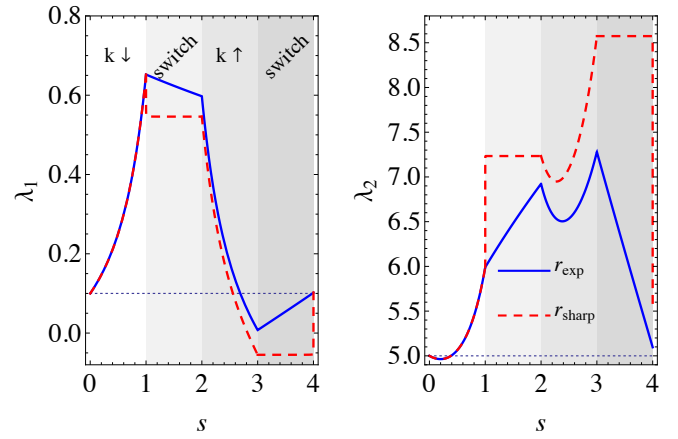


FIG. 6: Flow of the couplings  $\lambda_{1,2}$  along one closed-loop contour as defined in Fig. 5. The red dashed curve shows the switching from  $r_{\text{flat}}$  to  $r_{\text{sharp}}$  and the blue solid curve the switch from  $r_{\text{flat}}$  to  $r_{\text{exp}}$ . At the end of the closed-loop integration the values of the couplings  $\lambda_{1,2}$  deviate slightly from their initial values.

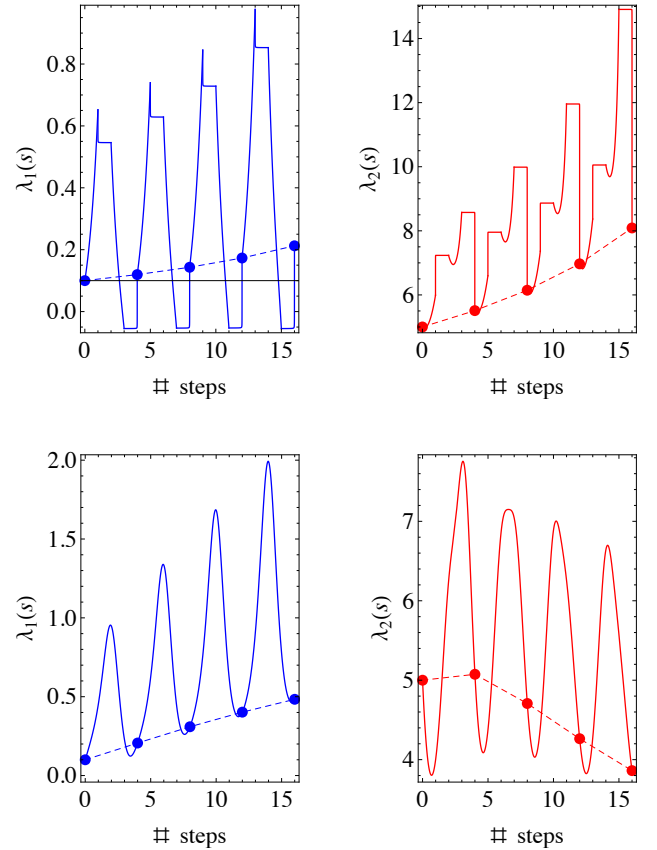


FIG. 7: Upper panels: Flow of the couplings  $\lambda_{1,2}$  along four rectangular closed loops switching from  $r_{\text{flat}}$  to  $r_{\text{sharp}}$ . Lower panels: Flow in  $k - R_k$  space with  $r_{\text{flat}}$  and  $r_{\text{sharp}}$  along an ellipse contour (counter clockwise), with an interpolation starting with scale  $k^{(\text{in})} = 2.0$  and  $r = r_{\text{flat}}(y) + r_{\text{sharp}}(y/2)$ , cf. App. F.



### III. OPTIMISATION

In order to obtain the best possible results from the functional renormalisation group approach within a given truncation we would like to single out the optimal regulator scheme for the underlying systematic expansion. Here we follow the setup of functional optimisation put forward in [7]. The discussion of systematic error estimates related to optimisation requires a norm on the space of theories (at  $k = 0$ ) in order to measure the severeness of the deviations. Here, we are not after a formal definition but rather a practical choice of such a norm.

We illustrate complications with the definition of such a norm by means of a simple example: we restrict ourselves to the local potential approximation (LPA), or LPA' where in the latter we take into account constant wave function renormalisations  $Z_k$ . Then, a seemingly natural choice is the cartesian norm on theory space spanned by the constant vertices  $\lambda_n = \Gamma^{(n)}[\phi_{\text{EoM}}]$  evaluated, e.g., at the equation of motion  $\phi_{\text{EoM}}$ . However, this falls short of the task as it weights a deviation in higher correlations or vertices  $\lambda_n$  in the same way as that of the lower ones, despite the fact that the lower ones are typically more important. Additionally, the  $\Gamma^{(n)}$  are neither renormalisation group invariant nor do they scale identically, see (A2).

If we extend the above setting to a general vertex expansion scheme, the coordinates in theory space are related to  $\Gamma^{(n)}[\phi_{\text{EoM}}](p_1, \dots, p_n)$ . These quantities are operators and the definition of the related  $n$ th axis of the coordinates system requires a suitably chosen operator norm, for a more detailed discussion see [34]. Even though this general case can be set-up, for most practical purposes it is sufficient to rely on a simple definition of a norm adapted to the approximation at hand.

Let us assume that we found a norm that allows to define the length  $L[\mathcal{C}]$  of a given flow along a trajectory  $\mathcal{C}$  in theory space parameterised with  $s \in [0, 1]$ , flowing from some regulator  $R_{s=0}$  to  $R_{s=1}$ . For example, we can consider the global flow with a given regulator from  $k = \Lambda$  to  $k = 0$ , i.e., the flow trajectory does not necessarily have to be a closed loop. The discussions in the previous section suggest that, in a given approximation, we should try to minimise this length in order to minimise the systematic error. Accordingly, we have to compare the lengths of different trajectories  $L[\mathcal{C}]$ . This heuristic argument can be made more precise, [34]: without approximation the final effective action  $\Gamma[\phi, R = 0]$  does not depend on the trajectory, in other words

$$\frac{\delta}{\delta R_s(p)} \int_0^1 ds \partial_s \Gamma[\phi, R] = 0. \quad (26)$$

Note however, that this discussion bears an intricacy, as it implies the comparison of the length of flow trajectories of physically equivalent effective actions  $\Gamma[\phi, R^A]$  and  $\Gamma[\phi, R^B]$  towards  $\Gamma[\phi, 0]$ . Therefore, we should compare trajectories that always start at physically equivalent

effective actions at a large physical cutoff scale.

#### A. Physical Cutoff Scale

The cutoff parameter  $k$  is usually identified with the physical cutoff scale, but such an identification falls short in the general case. To understand this, let us re-evaluate the example of the flows with  $r^A$  and  $r^B$  leading to the circular flow (12). In the spirit of the discussion above it seems to be natural to compare the two flows from  $k = \Lambda$  to  $k = 0$  with the regulators  $r_{s=0} = r^A$  and  $r_{s=1} = r^B$ , respectively, while the  $s$ -flow in this example simply switches the regulator at a fixed scale  $k = \Lambda$ . This picture fails trivially for

$$r_1(x) = r_0(x/\lambda), \quad \text{with} \quad R_{1,k} = R_{0,\lambda k}, \quad (27)$$

where the change of regulators simply amounts to changing the scale. As trivial as this example is, it highlights a key question:

*What is the physical cutoff scale for a given regulator?*

In Ref. [7] it has been argued that within practical applications it is suggestive to use the physical gap of the theory as the practical definition of the physical infrared or cutoff scale. Strictly speaking this asks for the evaluation of the poles and cuts of the theory in a real time formulation. For the scalar and Yukawa-type theories explicitly discussed in the present work it has been shown in [35] that the real-time pole masses and the imaginary time curvature masses are very similar in advanced approximations. For the present purpose these subtleties are not relevant and we define the inverse gap as the maximum of the imaginary time propagator,

$$\frac{1}{k_{\text{phys}}^2} = \max_{p, \phi} G[\phi, R]. \quad (28)$$

For the sake of simplicity we have restricted ourselves to constant backgrounds  $\phi$ . In the general case, (28) picks out the maximal spectral value of the propagator  $G$  [7]. Note also that in theories with several fields one has to monitor the gaps of all the fields involved. In the present work this is important within the example theories studied: the relativistic  $O(M) \oplus O(N)$  models as a simple model theory, as well as a non-relativistic Yukawa model for impurities in a Fermi gas. A further intricacy originates in different dispersion relations of the fields involved such as relativistic scalar field with  $p^2$  and fermionic fields with  $\not{p}$ . Then, the relative physical cutoff scale may involve a nontrivial factor in comparison to the gap. The latter subtlety will be discussed elsewhere.

Note also, that in (28) a fixed identical RG scheme for all regulators is required, as the propagator is not invariant under RG transformations  $\partial_s G[\phi, R] = -2\gamma_\phi G[\phi, R]$ , cf. App. A. Such a fixed RG scheme can be defined by first selecting one specific flow from  $k_{\text{phys}} = \Lambda$  to  $k_{\text{phys}}(k = 0)$ ,



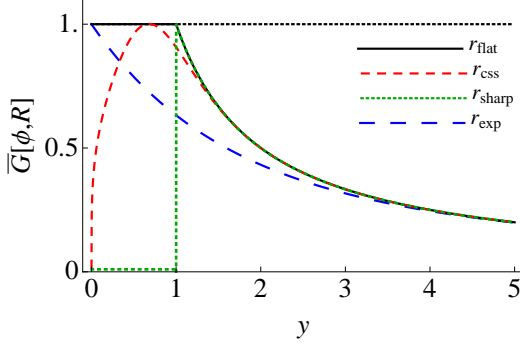


FIG. 8: Comparison of different normalized and regularized propagators  $y(1+r_i(y))^{-1}$  with shape functions  $r_i(y)$  as listed in Tab. I. Here,  $r_{\text{css}}$  is plotted with  $b = 1.25$ ,  $c = 3$  and  $h = 0.12$ .

the latter being the physical gap of the theory at  $k = 0$ . Then regulator changing flows such as defined in (10) at the fixed physical UV scale  $k_{\text{phys}} = \Lambda$  lead to initial conditions within the same RG scheme defined at  $k = 0$ . This leads to closed flows without taking into account a further RG transformation at  $k = 0$ . Hence, (28) implies that the normalised dimensionless propagator satisfies a renormalisation group invariant bound,

$$\bar{G}[\phi, R] \leq 1 \quad \forall p, \phi, \quad \text{with} \quad \bar{G}[\phi, R] = k_{\text{phys}}^2 G[\phi, R]. \quad (29)$$

In summary, we call theories in the presence of a regulator physically equivalent, if the gaps  $k_{\text{phys}}$  of all fields agree. In Fig. 8 we present some examples for this criterion for classical propagators in a theory with  $V''(\phi_{\text{min}}) = 0$ . These examples are relevant for the LPA approximation which we predominantly use in the present work.

### B. Optimisation and Length of a RG Trajectory

Now we are in the position to define the length of a flow trajectory  $\mathcal{C}$ . Keeping in mind the discussion of the coordinate system in theory space at the beginning of this Section, we reduce the task by using the effective action itself, evaluated on fields close to the solution of the quantum equations of motion  $\phi_{\text{min}}$  with

$$\left. \frac{\delta \Gamma[\phi, R]}{\delta \phi} \right|_{\phi=\phi_{\text{min}}} = 0. \quad (30)$$

The value of the effective action has no physics interpretation and depends on the renormalisation procedure, i.e., the chosen regulator and initial condition. Therefore, we resort to the second derivative  $\Gamma^{(2)}[\phi, R]$  rather than to  $\Gamma[\phi, R]$  itself. Indeed, the natural choice is the connected two-point correlation function or rather the normalised dimensionless two-point function  $\bar{G}[\phi, R] = k_{\text{phys}}^2 \langle \phi(p)\phi(-p) \rangle_c$ , cf. (29). Here, the subscript  $c$  refers to the connected part. This is motivated by the fact that the master equation (2) only depends on the propagator,

as do the operator representations for the total  $t$ - and  $s$ -derivatives, (14) and (B2).

Measuring the length of the flow of the dimensionless propagator  $\bar{G}[\phi, R]$  requires a coordinate system in theory space where the axes are, e.g., expansion coefficients of the propagator,  $\bar{G}^{(n)}$  or the spectral values of the propagators linked to an expansion in the eigenbasis of  $\bar{G}$

$$\bar{\lambda}_{\text{max}} = 1, \quad \text{for} \quad \bar{\lambda} \in \text{spec } \bar{G}[\phi, R(k_{\text{phys}})]. \quad (31)$$

To summarise, the procedure is choosing an operator norm  $\|\cdot\|$  for  $\bar{G}$  as well as for  $\partial_s \bar{G}$ . Then, we define the length of a flow trajectory  $\mathcal{C}$  with that of the length of  $\|\partial_s \bar{G}\|$ .

Before we come to integrated flows, let us evaluate the consequences of the discussion above. Firstly, we assert that monotonous flows are shorter than non-monotonous ones. Assuming already a restriction to monotonous flows for  $G$  and hence  $\bar{G}$ , we find a simple optimisation criterion in terms of the dimensionless propagator:  $\bar{G}$  is bounded from above by unity, see (29). Moreover, for optimal regulators the propagator is already as close as possible to this bound due to its monotonous dependence on  $t$ . This leaves us with

$$\begin{aligned} \|\bar{p}^2 (\bar{G}[\phi, R_{\text{opt}}] - \bar{G}[\phi, 0])\| \\ = \min_{R \in R(k_{\text{phys}})} \|\bar{p}^2 (\bar{G}[\phi, R] - \bar{G}[\phi, 0])\|, \end{aligned} \quad (32)$$

where  $\bar{p}^2 = p^2/k_{\text{phys}}^2$ . The prefactor  $\bar{p}^2$  has been introduced for convenience to easily accommodate also for massless modes at vanishing cutoff scale. The criterion (32) has been derived in [7], where it also has been shown that for optimised regulators local integrability is restored.

With Eq. (32), for a given background  $\phi$ , we have reduced theory space to a one-dimensional subspace with a simple cartesian norm. Still, the space of regulators is infinite-dimensional and the length of a given flow curve parametrised by  $s$  is related to the size of the flow operator equations (14) and (B2) for  $t$ -flows or  $s$ -flows, respectively. The flow operators involve second-order  $\phi$  derivatives as well as kernel of the flow operator,

$$\mathcal{K}[\phi, R] = G[\phi, R] \partial_s R G[\phi, R]. \quad (33)$$

The  $\phi$  derivatives act on the complete set of observables and their action is general. Therefore, we simply have to integrate the size of  $\mathcal{K}[\phi, R]$  along the flow for computing a relevant length. For constant backgrounds  $\phi$  we integrate over all spectral values of the operator

$$\|\mathcal{K}[\phi, R]\| = \int_0^\infty dp^2 |\mathcal{K}[\phi, R]|, \quad (34)$$

giving a dimensionless quantity. This spectral definition can be extended to general backgrounds. Moreover, it can be extended to more general norms that, e.g., take into account the importance of smooth regulators for the

derivative expansion [7]. The norm in (34) diverges for  $\mathcal{K}[\phi, R]$  showing an infrared singularity with more than  $1/p^2$ . This can be amended with additional powers of  $\bar{p}^2$ .

In summary this leads us to the final expression for the length of a trajectory at a given value of  $V''(\phi)$ ,

$$L[V'', R] = \int_0^1 ds \sqrt{1 + \|\mathcal{K}[\phi(V''), R]\|^2}, \quad (35)$$

where  $\phi(V'')$  is chosen such that  $V''(\phi(\omega))$  is fixed. Then, (35) is the length of the trajectory for  $G$ ,

$$\int_0^1 \sqrt{ds^2 + \|dG[\phi(V''), R]\|^2} = L[V'', R], \quad (36)$$

where we have used that  $\partial_s V'' \equiv 0$ . With (36) the optimisation criterion (32) now can be recast into

$$L[V'', R_{\text{opt}}] = \min_{R \in R(\Lambda_{\text{phys}})} L[V'', R], \quad (37)$$

where  $\Lambda_{\text{phys}}$  indicates that all flows start at the same physical scale. Note that identical physical UV scales are typically easily identified. Hence, for global flows from the ultraviolet to the infrared we have trajectories with  $\Gamma_\Lambda = \Gamma[\phi, R_\Lambda]$  with  $R(s=0) = R_\Lambda$ , and  $\Gamma_{k=0} = \Gamma[\phi, 0]$  with  $R(s=1) \equiv 0$ . The optimal regulator should minimise the length of the flow  $L[V'', R]$  for all  $\phi$ . A comparison of the length for different regulators will be presented in the following Section III C.

We close with the remark that both criteria, (32) and (37), implement the functional optimisation criterion from [7], and hence are identical. In practical applications the one or the other may be more easily accessible.

### C. Practical Optimisation

Let us exemplify the above construction at the example of the LPA approximation for one real scalar field. Its propagator for a given gap  $k_{\text{phys}}$  reads

$$G[\phi_{\text{min}}, R] = \frac{1}{p^2 + \omega_{\text{min}} + R}. \quad (38)$$

where it is understood that the cutoff scales in the regulator  $R$  is adjusted such that the maximum of the propagator is  $1/k_{\text{phys}}^2$ , and  $\omega_{\text{min}} = V''(\phi_{\text{min}})$  stands for the curvature at the minimum of the effective potential. Now, we use that an optimised regulator minimises infinitesimal flows as well as the rest of the flow towards  $k=0$ . This statement holds for correlation functions and, in particular, for the propagator entailing that the difference between the optimal propagator for a given physical cutoff scale  $k_{\text{phys}}$  and the propagator at  $k=0$  is minimal.

Let us assume for the moment that  $\omega_{\text{min}}$  is already at the value it acquires at  $k=0$ . Then, we are left with the

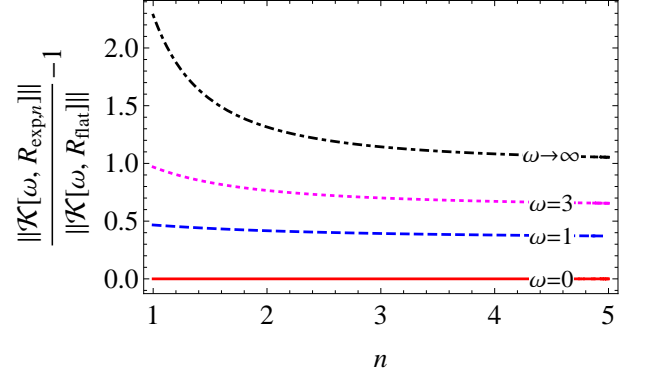


FIG. 9: Deviation of the ratio of  $\|\mathcal{K}[\omega, R_{\text{exp}, n}]\|/\|\mathcal{K}[\omega, R_{\text{flat}}]\| - 1$ , as a function of the power  $n$  in the exponential, for  $\omega \in [0, \infty)$ . For  $\omega \rightarrow \infty$  the bound saturates. For  $\omega = 0$  the norm is a total derivative and is independent of the choice of regulator, hence the ratio is unity:  $\omega = 0$ : red straight line,  $\omega = 1$ : blue dashed line,  $\omega = 3$ : magenta dotted line,  $\omega = \infty$ : black dashed-dotted line.

condition to minimise

$$\begin{aligned} & |G[\phi_{\text{min}}, R] - G[\phi_{\text{min}}, 0]| \\ &= \left| \frac{R}{(p^2 + \omega_{\text{min}} + R)(p^2 + \omega_{\text{min}})} \right|, \end{aligned} \quad (39)$$

for all momenta with the constraint (28). We now make a further simplification and set  $\omega_{\text{min}} = 0$ . Then, we are left with minimising

$$\left| \frac{r}{(p^2 + R)} \right|, \quad (40)$$

for all momenta. For momenta  $p^2 \geq k_{\text{phys}}^2$  we immediately arrive at  $r_{\text{opt}} = 0$ . For  $p^2 < k_{\text{phys}}^2$  the regulator has to be positive in order to account for the gap condition (28). If this condition is saturated, (39) is minimised, leading to  $p^2 + p^2 r_{\text{opt}} = k_{\text{phys}}^2$ , and hence  $r_{\text{opt}} = k_{\text{phys}}^2 / (p^2 - 1)$  for the momenta  $p^2 < k_{\text{phys}}^2$ . In combination with the vanishing for  $p^2 \geq k_{\text{phys}}^2$  this leads to the unique optimised regulator in LPA,

$$r_{\text{opt}} = \left( \frac{k_{\text{phys}}^2}{p^2} - 1 \right) \theta(k_{\text{phys}}^2 - p^2), \quad (41)$$

the flat or Litim regulator [22–24]. Note that there it has been introduced as one of a set of optimised regulators, being distinct by its analytic properties. It has been singled out as the unique solution of the functional optimisation in Ref. [7]. Indeed, the critical exponents in  $O(N)$  theories truncated in a local potential approximation with this regulator are closest to the physical ones. The above simplified derivation can be upgraded to also take into account a given fixed  $\omega_{\text{min}}$ . Again this leads to (41) with  $k_{\text{phys}}^2 \rightarrow k^2$  where  $k^2$  runs from  $k_{\text{phys}}^2 - \omega_{\text{min}}$  to zero.

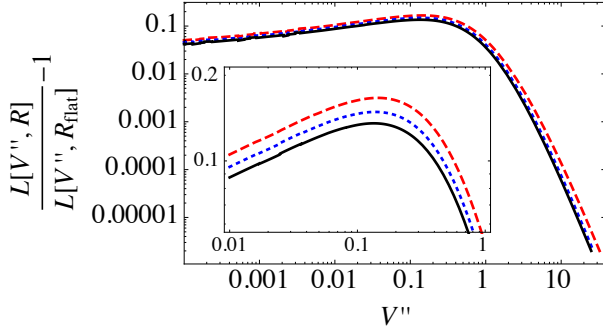


FIG. 10: Length  $L[V'', R]/L[V'', R_{\text{flat}}] - 1$  for different exponential regulators in comparison to the flat one:  $n = 1$ : red dashed line,  $n = 2$ : blue dotted line,  $n = 4$ : black straight line. The length is minimised for the flat regulator.

Let us now also compare the lengths of the trajectories as defined in Sec. III B. The definition was adjusted such that it does not require the knowledge about  $\Gamma^{(2)}[\phi, R]$  along the flow leading to simple practical computations. A more elaborate version of the present case is straightforwardly implemented by relaxing  $\partial_s V'' \neq 0$ .

In Fig. 9 we first compare the norms, (34), of the flow operator for different values of  $\omega = V''/k^2$ . We show the deviation of the ratios  $\|\mathcal{K}[\omega, R_{\text{exp}, n}]/\|\mathcal{K}[\omega, R_{\text{flat}}]\| - 1 \geq 0$  from one for all values of  $\omega$ . Here,  $R_{\text{exp}, n}$  are the exponential regulators with the corresponding shape function

$$r_{\text{exp}, n}(y) = \frac{y^{n-1}}{e^{y^n} - 1}, \quad (42)$$

which is a specific subclass of the exponential interpolating regulator in Tab. I with  $a = 1, b = 0$ . The deviation is always bigger than zero, which singles out the flat regulator as the optimal one in LPA, see Fig. 9. Note in this context that Fig. 9 gives us the full information of the relative size of the integrands in the length of the flow in (35): for a given  $V'' > 0$  the related  $\omega$  diverges with  $1/k_{\text{phys}}^2$ . This is easily confirmed with the explicit computation of the length, summarised in Fig. 10, where the global length is shown for given  $V''$ . One can observe that the flat regulator minimises the flow length  $L[V'', R]$  which supports the optimisation criterion developed in Sec. III B.

Note also that the norms are defined such that the information about the physical scales  $k_{\text{phys}}^2 (s = 1)/k_{\text{phys}}^2 (s = 0)$  is only encoded in  $\omega(s = 1)/\omega(s = 0)$ . Hence, for large  $V''$  in comparison to the physical cutoff scales the difference between the different flows is large. However, in this regime the absolute size of the flow is small.

#### IV. CRITICAL BEHAVIOR OF MULTI-FIELD MODELS

Many interesting systems include a collection of different field degrees of freedom. In this situation the choice of

suitable combinations of regulators is not straightforward and we have already mentioned the relativistic Yukawa models with structurally different dispersion for scalars and fermions. Here we study this case within a simple situation, a bosonic  $O(M) \oplus O(N)$  model. We show that the choice of relative cutoff scales generally has a crucial impact on the obtained results for the critical physics: The  $O(M) \oplus O(N)$  model has two competing order parameter fields and the competing order makes it particularly sensitive to small effective changes of the relevant parameters. The model is studied in the lowest order of the derivative expansion, in LPA. It is well known that such a truncation already captures well the critical physics of scalar models despite the lack of non-trivial momentum dependences of propagators and vertices. The latter encode the anomalous dimensions of the system which are quantitatively small, here, and hence can be neglected.

However, the momentum dependences are also important for taking into account the momentum transfer present in the diagrams on the right hand side of the flow equation. For identical physical cutoff scales this momentum transfer is minimised. In turn, for shifted relative physical cutoff scales of different fields the diagrams have a sizeable momentum transfer. In such a case, physics that is well incorporated in the LPA with identical physical cutoff scales, is lost if the difference between the physical cutoff scales grows large. If one goes beyond LPA within systematic momentum-dependent approximation schemes this relative cutoff scale dependence eventually disappears. The discussion also emphasises the necessity of identical physical cutoff scales within a given approximation in the sense of an optimisation of approximation schemes.

In the present section we highlight the physics changes that are triggered by the change of relative cutoff scales in LPA. As discussed above, due to the missing momentum dependences of LPA, different relative cutoff scales effectively lead to different actions at  $k = 0$ , see also Fig. 3. In LPA, the bosonic,  $d$ -dimensional  $O(M) \oplus O(N)$  model has the following effective action, [36–38],

$$\Gamma_k = \int_x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 + U_k(\phi, \chi) \right], \quad (43)$$

where  $\phi$  and  $\chi$  are  $N$ - and  $M$ -component fields, respectively. The effective potential  $U_k(\bar{\rho}_\phi, \bar{\rho}_\chi)$  only depends on the field invariants  $\bar{\rho}_\phi = \phi^2/2$  and  $\bar{\rho}_\chi = \chi^2/2$ . The scale-dependent dimensionless effective potential is given by

$$u = u(\rho_\phi, \rho_\chi) = k^{-d} U_k(\bar{\rho}_\phi, \bar{\rho}_\chi) \quad \text{with} \quad \rho_i = k^{2-d} \bar{\rho}_i, \quad (44)$$

and  $i \in \{\phi, \chi\}$ . We further introduce the shape functions  $r_\phi$  and  $r_\chi$  to regularise the  $\phi$  and  $\chi$  field modes, respectively. The flow equation for the dimensionless effective

potential (44) reads

$$\begin{aligned} \partial_t u = & -d u + (d-2)\rho_\phi u^{(1,0)} + (d-2)\rho_\chi u^{(0,1)} \\ & + I_{R,\phi}(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) + (N-1)I_{G,\phi}(u^{(1,0)}) \\ & + I_{R,\chi}(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) + (M-1)I_{G,\chi}(u^{(0,1)}), \quad (45) \end{aligned}$$

where we have introduced suitable threshold functions  $I_{i,j}(x, y, z)$ ,  $i \in \{R, G\}$ ,  $j \in \{\phi, \chi\}$  to separate the loop integration over the radial and Goldstone modes for the two fields. The explicit expressions for these threshold functions are listed in App. D. The arguments of the threshold functions are given by  $\omega_\phi = u^{(1,0)} + 2\rho_\phi u^{(2,0)}$ ,  $\omega_\chi = u^{(0,1)} + 2\rho_\chi u^{(0,2)}$  and  $\omega_{\phi\chi} = 4\rho_\phi\rho_\chi(u^{(1,1)})^2$ . For calculations, we expand the effective potential about the flowing minimum  $(\kappa_\phi, \kappa_\chi)$ , to wit

$$u(\rho_\phi, \rho_\chi) = \sum_{1 \leq i+j \leq \text{ord}} \frac{\lambda_{ij}}{i!j!} (\rho_\phi - \kappa_\phi)^i (\rho_\chi - \kappa_\chi)^j. \quad (46)$$

In this truncation we follow the flow of the couplings  $\kappa_i$  and  $\lambda_{ij}$  which are given in App. E. These  $O(N) \oplus O(M)$  models possess a rich variety of fixed points exhibiting different types of multi-critical behaviour relevant to a number of physical systems [39–42]. For our further investigations, we list the properties of the three most important fixed points:

(i) The *decoupled fixed point* (DFP) is characterised by a decomposition into two disjoint  $O(N)$  and  $O(M)$  models where all mixed interactions vanish, e.g.,  $\lambda_{11} = 0$ . It inherits all of the properties of the Wilson-Fisher (WF) fixed points of the separate sub-sectors.

(ii) The *isotropic fixed point* (IFP) features a symmetry enhancement where at each order in the fields the couplings are degenerate, e.g.,  $\lambda_{20} = \lambda_{02} = \lambda_{11}$ . Therefore, the fixed point coordinates agree with the ones from an  $O(N+M)$  symmetric model and it inherits all of its critical exponents.

(iii) The *biconical fixed point* (BFP) is a non-trivial fixed point with interactions in both sectors that does not provide a symmetry enhancement. This fact makes it interesting for our further analysis, because it can be easily distinguished by means of the critical exponents of a single field model.

We have listed values for the largest critical exponent  $y_1 = 1/\nu_1$  for the models and fixed points which are important to this work in Tab. II, showing results for different levels of the truncation. Generally, we sort the critical exponents according to the definition  $y_1 > y_2 > y_3 > \dots$ .

### A. Fixed Points and Relative Cutoff Scales

In this section, we examine the effects that occur when dealing with models whose sub-sectors are defined on

Fixed point	$\rho^2$	$\rho^3$	$\rho^4$	$\rho^5$	$\rho^6$
BFP in $O(2) \oplus O(2)$	1.78	1.61	1.78	1.77	1.75
BFP in $O(1) \oplus O(2)$	2.16	1.42	1.65	1.63	1.62
WF in $O(2)$	1.61	1.32	1.40	1.42	1.41
WF in $O(1)$	2.00	1.37	1.54	1.55	1.54

TABLE II: List of values for the largest critical exponent  $y_1$  for selected fixed points in LPA to ascending order in the truncation.

separate cutoff scales. To that end, we investigate the  $O(M) \oplus O(N)$  model using flat regulators in the two sectors, however, with separated cutoff scales

$$\begin{aligned} r_\phi(y) &= \left(\frac{1}{y} - 1\right) \theta(1-y), \\ r_\chi(y) &= r_\phi\left(\frac{y}{c}\right) = \left(\frac{c}{y} - 1\right) \theta\left(1 - \frac{y}{c}\right). \quad (47) \end{aligned}$$

Consequently, for a generic  $c \neq 1$  the fluctuations of one field are integrated out earlier than the fluctuations of the other: The second regulator has a built-in shift of all scales  $k^2 \rightarrow ck^2$ ,  $\Lambda^2 \rightarrow c\Lambda^2$ . For  $c < 1$  this leads to a suppression of the  $\chi$  sector and the RG flow does not experience any  $\chi$  fluctuations. Inversely,  $c > 1$  suppresses the  $\phi$  sector in a similar way. Only for  $c = 1$  the physical cutoff scales are identical. Note that the statement about identical physical cutoff scales  $k_{\text{phys}}(\phi) = k_{\text{phys}}(\chi)$  is only trivial in the present case where  $\phi$  and  $\chi$  are both scalar fields with the same dispersion and interactions. In the general case it is non-trivial to identify the relative cutoff scales and where the different representations of the optimisation may pay-off in particular.

The threshold functions for this choice of regulators with separate cutoff scales can be found in App. D. Now, we discuss the dependence of critical exponents of different fixed points on changes of the relative cutoff scale.

*DFP Critical Exponents* At the DFP the fields  $\phi$  and  $\chi$  decouple. Introducing a scale-shifted regulator in a single  $O(N)$ -model, does not induce a difference in the critical exponents, since every momentum can simply be rescaled. The results of our investigation for the DFP in the  $O(2) \oplus O(1)$  model are displayed in the left panel of Fig. 11, confirming the previous statement. The critical exponents for all values of  $c$  can be identified with the Wilson-Fisher critical exponents Tab. II at the corresponding order of the truncation.

*IFP Critical Exponents* Introducing  $c \neq 1$  revokes the symmetry between  $\phi$  and  $\chi$ , destroying the key property of this fixed point. Therefore, we will not further investigate this fixed point in the present context. However, we note that this is an illustrative example for how a unphysical regulator choice can lead to artificial regulator dependencies of FRG predictions.

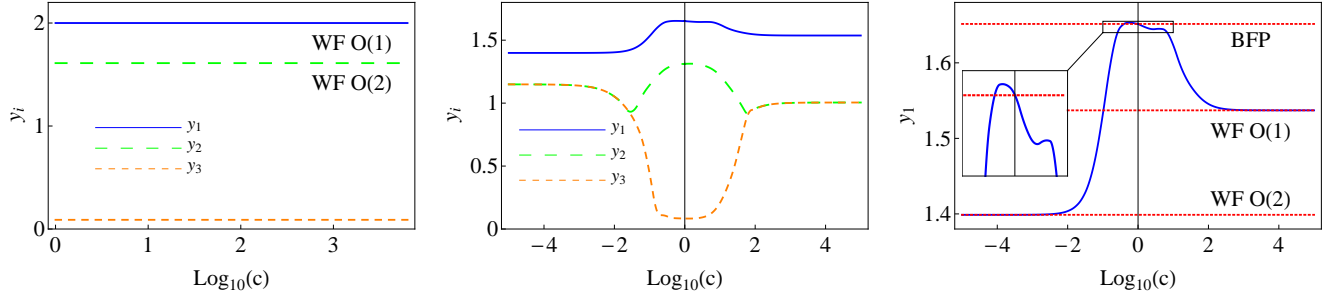


FIG. 11: Left panel: DFP critical exponents of the  $O(2) \oplus O(1)$  model in LPA to order  $\rho^2$  using separated cutoff scales. Due to the complete decoupling of both sectors, no dependence on the relative cutoff scale  $c$  can be observed. Middle panel: BFP critical exponents of the same model in LPA to order  $\rho^4$ . Here, the two sectors are coupled showing a severe dependence on  $c$ . Right panel: Close-up of the largest BFP critical exponent  $y_1$  of the  $O(2) \oplus O(1)$ -model in LPA to order  $\rho^4$  exhibiting the limiting cases  $c \ll 1$  and  $c \gg 1$  where one of the sectors is effectively suppressed and the critical behaviour is described by the Wilson-Fisher FP of the other sector.

**BFP Critical Exponents** New insights can be gained by looking at the  $c$  dependence of the BFP where both sub-sectors are explicitly coupled. The critical exponents of the BFP clearly exhibit a severe dependence on the relative scale factor  $c$ , see middle and right panel of Fig. 11 for the example of the  $O(2) \oplus O(1)$  model. In fact, the largest critical exponent,  $y_1$ , of the BFP tends to the WF critical exponents of one of the single sub-sectors as the other one is suppressed by a large relative cutoff scale. Explicitly, for  $c \ll 1$ , the critical exponent  $y_1$  of the  $O(2) \oplus O(1)$  BFP approaches the value of the WF fixed point of the  $O(2)$  model. Analogously, for  $c \gg 1$ ,  $y_1$  approaches the WF fixed point of  $O(1)$  model.

We assert that in systems with various field degrees of freedom, the choice of their relative cutoff scales has a severe impact on the described physics in LPA due to the missing momentum dependences. The change of the ratio  $c$  of the cutoff scales induces a change of universality classes. In the present simple  $O(M) \oplus O(N)$  models the generic choice is  $c = 1$  as the fields involved have identical dispersions and interactions. We also emphasise again, that in more complicated systems with different sectors, and in particular fermion-boson systems, there is no clear *a priori* criterion for a suitable choice of regulators and their relative cutoff scales, see also the following section. In the inset of the right panel of Fig. 11, we further show that the value of the critical exponent  $y_1$  at  $c = 1$  is not singled out as a local extremum of the critical exponent  $y_1(c)$ . We suggest that a control of this issue in multi-field models can be gained by the practical optimisation procedure presented in Sec. III which, however, is beyond the scope of the present work.

## B. Truncation Dependence

The integrability condition (12) fails as we truncate the effective action. In turn, this suggests that the regulator dependence of the results becomes weaker when the level of truncation is increased. In this section we examine

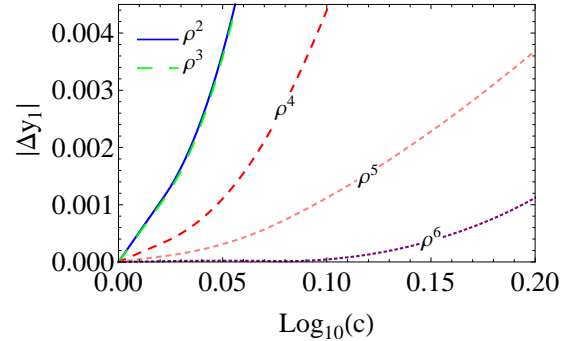


FIG. 12: Deviation from the critical exponent  $y_1$  in the  $O(2) \oplus O(2)$  model in LPA from order  $\rho^2$  to order  $\rho^6$  as a function of the relative cutoff scale  $c$ . We clearly observe a flattening of the curves for higher orders of the truncation.

the dependence of the BFP critical exponent  $y_1$  on the relative cutoff scale factor  $c$  as a function of the level of truncation. We focus on the  $O(2) \oplus O(2)$  model and compare different orders of the LPA, i.e., to the orders between  $\rho^2$  up to  $\rho^6$ .

Fig. 12 shows the deviation of  $y_1$  at a given  $c > 1$  from the value at  $c = 1$  weighted by its difference to the limiting case of the corresponding  $O(2)$  critical exponent

$$|\Delta y_1| = \left| \frac{y_1(c) - y_1(1)}{y_1(1) - y_{1,O(2)}} \right|. \quad (48)$$

We see that for increasing order in the LPA from  $\rho^2$  up to  $\rho^6$ , the dependence of  $y_1$  on the relative cutoff scale  $c$  becomes weaker and weaker as suggested by the consecutive flattening of the curves. We conclude that a better truncation is more robust to uneducated regulator choices or in other words a low-level truncation requires a more sophisticated choice of the regulator scheme.

## V. NON-RELATIVISTIC FERMION-BOSON MODELS

In the last Section IV, we have discussed the question of relative cutoff scales in a simple scalar model with identical dispersions and interactions for the different fields. In the present Section we discuss relative cutoff scales and the impact of the shape dependence of regulators in a more complicated situation of a non-relativistic Yukawa system describing fermionic atoms and molecules. In contradistinction to the LPA approximation used in the last Section we also take into account momentum and frequency dependences of the propagators. Naturally, this does not fully cure the lack of momentum dependences of the approximation and we expect a modest regulator dependence of the corresponding results. Further aspects in non-relativistic systems are  $N$ -body hierarchies which lead to complete resummation schemes for three-, four- and  $N$ -body systems, which has been worked out for fermionic three-, and four-body cases for various systems, [43–48]. This gives us further access for an assessment of the regulator dependence of our results.

The FRG has found multiple applications in the study of non-relativistic systems, ranging from few- [43, 44, 49] to many-body problems [50–55]. In a prototypical scenario in condensed matter physics, fermionic and bosonic degrees of freedom interact with each other. Such situations occur for instance in models which describe the formation of molecules from atoms. Other examples include the interaction of electrons with collective excitations, such as phonons or magnons. In addition to the question of equivalent cutoff scales of fermions and bosons, similarly to the coupled  $O(N)$  models discussed in Section IV, the exact fermionic  $N$ -body hierarchies present an additional challenge for the mixed non-relativistic bosonic-fermionic system within the evaluation of FRG flows. Optimal approximations have to take into account these exact hierarchies in addition to taking care of the momentum transfer. If both properties cannot be rescued in a given approximation, it is a priori not clear, in which order the various fields have to be integrated out for optimal results. Hence, in this situation, the question of the optimal ratio of cutoff scales is even more complicated as in the bosonic example treated in Section IV.

In the present Section we do not aim at a full resolution of this intricate question, but rather highlight the ensuing difficulties. We provide an analysis of the regulator dependences arising in a system consisting of a single impurity immersed in a non-relativistic Fermi sea of atoms at zero temperature. In this so-called Fermi polaron problem [56–59] the interaction of the impurity  $\psi_\downarrow$  with the fermions  $\psi_\uparrow$  in the Fermi sea is determined by the exchange of a molecular field  $\phi$  which represents a bound state of the  $\downarrow$ -impurity with one of the medium

$\uparrow$ -atoms. The system is described by the action

$$S = \int_{\mathbf{x}, \tau} \left\{ \sum_{\sigma=\uparrow, \downarrow} \psi_\sigma^* [\partial_\tau - \Delta - \mu_\sigma] \psi_\sigma + \phi^* [\partial_\tau - \Delta/2 + \nu_\phi] \phi + h(\psi_\uparrow^* \psi_\downarrow^* \phi + h.c.) \right\}, \quad (49)$$

where  $\int_{\mathbf{x}, \tau} = \int d^3x d\tau$ ,  $\Delta$  is the Laplace operator and we suppressed the arguments  $\mathbf{x}, \tau$  of the fields. Furthermore the Grassmann-valued, fermionic fields  $\psi_\uparrow$  and  $\psi_\downarrow$  represent  $\uparrow$ - and  $\downarrow$ -spin fermions of equal mass  $m$ . Note that we work in units  $\hbar = 2m = 1$  and  $\sigma = (\uparrow, \downarrow)$ . The associated chemical potentials  $\mu_\sigma$  are adjusted such that the  $\uparrow$ -fermions have a finite density  $n_\uparrow = k_F^3/(6\pi^2)$ , with  $k_F$  the Fermi momentum, while there is only a single  $\downarrow$ -atom. In this limit the action Eq. (49) describes the problem of a single impurity immersed in a Fermi sea. The detuning  $\nu_\phi$ , together with the coupling  $h$  determines the interaction strength between the  $\downarrow$ - and  $\uparrow$ -atoms which is mediated by the exchange of the field  $\phi$ .

The impurity is dressed by fluctuations in the fermionic background. It becomes a quasi-particle, the Fermi polaron, which is characterised by particle-like properties such as an energy  $E_p$ , and a quasi-particle weight  $Z_p$ . The quasiparticle properties depend on the interaction between the impurity and the Fermi gas. Due to the presence of bound states, this interaction cannot be described within perturbation theory and requires non-perturbative approximations. Hence, it presents an ideal testing ground for methods such as the FRG.

In the following the prediction of  $E_p$  will serve as our observable to study the regulator dependences occurring in the RG evaluation of the model Eq. (49). The Fermi polaron problem is particularly interesting for our study since accurate numerical predictions for various quantities exist based on a bold diagrammatic Monte Carlo scheme [56]. For instance at unitarity, where the infrared scattering amplitude at zero scattering momentum – given by the scattering length  $a_s$  – diverges,  $k_F a_s \rightarrow \infty$ , the ground state energy is predicted to approach the value of  $E_p = -0.615 \epsilon_F$  [56], with  $\epsilon_F$  the Fermi energy.

We note that also other non-relativistic systems of coupled bosons and fermions are described by Eq. (49). For instance for a chemical potential  $\mu_\downarrow > -E_p$  the system exhibits the BEC-BCS crossover at low temperature as interactions are varied [60–62]. This crossover has been studied extensively by FRG methods [51, 52, 63–65].

### A. Truncation and Flow Equations

In the following we will solve the FRG flow equation (2) for the truncation of the effective action

$$\Gamma_k = \int_{\mathbf{p}, \omega} \left\{ \psi_\uparrow^* [-i\omega + \mathbf{p}^2 - \mu_\uparrow] \psi_\uparrow + \psi_\downarrow^* G_{\downarrow, k}^{-1}(\omega, \mathbf{p}) \psi_\downarrow + \phi^* G_{\phi, k}^{-1}(\omega, \mathbf{p}) \phi \right\} + \int_{\bar{\mathbf{x}}, \tau} h(\psi_\uparrow^* \psi_\downarrow^* \phi + h.c.), \quad (50)$$



where  $\int_{\mathbf{p},\omega} = \int \frac{d^3p}{(2\pi)^3} \int \frac{d\omega}{2\pi}$ . In this truncation the only RG scale  $k$  dependent quantities are  $G_{\downarrow,k}$  and  $G_{\phi,k}$ . While in previous work the flow of fully momentum dependent propagators  $G_{\downarrow,k}$  and  $G_{\phi,k}$  has been considered [54], we study here the regulator dependences arising in a derivative expansion where

$$\begin{aligned} G_{\downarrow,k}^{-1}(\omega, \mathbf{p}) &= S_{\downarrow}[-i\omega + \mathbf{p}^2] + m_{\downarrow}^2, \\ G_{\phi,k}^{-1}(\omega, \mathbf{p}) &= S_{\phi}[-i\omega + \mathbf{p}^2/2] + m_{\phi}^2, \end{aligned} \quad (51)$$

with scale-dependent wave function renormalisations  $S_{\downarrow}, S_{\phi}$ . It has been shown in [35] for relativistic Yukawa systems that this approximation of the full frequency- and momentum-dependence already captures well the full dependence. We expect this to be also the case in the present non-relativistic case. The RG scale dependent coupling constants  $m_{\downarrow}^2$  and  $m_{\phi}^2$  are related to the flowing static self-energies  $\Sigma_{\downarrow,\phi}(0, \mathbf{0})$ , e.g.  $m_{\downarrow}^2 = -\mu_{\downarrow} - \Sigma_{\downarrow}(0, \mathbf{0})$ . In the impurity problem the majority fermions are not renormalized,  $S_{\uparrow} = 1$ , and the density of the Fermi sea is determined by the chemical potential  $\mu_{\uparrow} = \varepsilon_F = k_F^2$ . In summary, from this truncation, we obtain the four flow equations, cf. App. H,

$$\begin{aligned} \partial_t m_{\phi}^2 &= \frac{h^2}{2\pi^2} \int_{k_F}^{\infty} dp \frac{p^2(\partial_t R_{\downarrow} + S_{\downarrow}\partial_t R_{\uparrow})}{[m_{\downarrow}^2 + R_{\downarrow} + S_{\downarrow}(2p^2 - \mu_{\uparrow} + R_{\uparrow})]^2} \\ \partial_t S_{\phi} &= -\frac{h^2}{\pi^2} \int_{k_F}^{\infty} dp \frac{S_{\downarrow}p^2(\partial_t R_{\downarrow} + S_{\downarrow}\partial_t R_{\uparrow})}{[m_{\downarrow}^2 + R_{\downarrow} + S_{\downarrow}(2p^2 - \mu_{\uparrow} + R_{\uparrow})]^3} \\ \partial_t m_{\downarrow}^2 &= \frac{h^2}{2\pi^2} \int_0^{k_F} dp \frac{p^2(\partial_t R_{\phi} - S_{\phi}\partial_t R_{\uparrow})}{[m_{\phi}^2 + R_{\phi} - S_{\phi}(p^2/2 - \mu_{\uparrow} + R_{\uparrow})]^2} \\ \partial_t S_{\downarrow}^2 &= -\frac{h^2}{\pi^2} \int_0^{k_F} dp \frac{S_{\phi}p^2(\partial_t R_{\phi} - S_{\phi}\partial_t R_{\uparrow})}{[m_{\phi}^2 + R_{\phi} - S_{\phi}(p^2/2 - \mu_{\uparrow} + R_{\uparrow})]^3}. \end{aligned} \quad (52)$$

We study the dependence of the predictions from the FRG using a continuous set of regulators  $R_{\downarrow,\uparrow}$  and  $R_{\phi}$  which are dependent on various parameters. We choose

$$\begin{aligned} R_{\phi}^k(p) &= c_{\phi} \frac{S_{\phi}k^2}{2} (a_{\phi} - b_{\phi}y) \frac{y^{n_{\phi}}}{e^{y^{n_{\phi}}} - 1} \\ R_{\downarrow}^k(p) &= c_{\downarrow} S_{\downarrow} k^2 (a_{\downarrow} - b_{\downarrow}y) \frac{y^{n_{\downarrow}}}{e^{y^{n_{\downarrow}}} - 1} \\ R_{\uparrow}^k(p) &= c_{\uparrow} S_{\uparrow} k^2 (a_{\uparrow} - b_{\uparrow}y) \sigma(p^2 - \mu_{\uparrow}) \frac{y^{n_{\uparrow}}}{e^{y^{n_{\uparrow}}} - 1} \end{aligned} \quad (53)$$

where  $y \equiv p^2/(c_i k^2)$  and  $\sigma(x) = 1, (-1)$  for  $x > 0$  ( $x \leq 0$ ) for the impurity  $\psi_{\downarrow}$  and boson field  $\phi$ . These regulators are similar to the regulators studied in the relativistic models in the previous sections, cf. Tab. I. Note however that for the bath fermions the pole structure due to the Fermi surface has to be accounted for so that here  $y \equiv (|\mathbf{p}^2 - \mu_{\uparrow}|)/(c_{\downarrow} k^2)$ . Similar to the definitions used in

Section IV, the parameters  $c_i$  ( $i = \phi, \uparrow, \downarrow$ ) allow for the study of changing the relative scales at which the various field are integrated out, while the other parameters allow for deformations of the regulator shape, cf. Fig. 8.

## B. Initial Conditions

As discussed in Section II A, first the initial values at the UV scale  $k = \Lambda$  have to be set. The initial value of  $m_{\phi}^2$  is determined by the interaction strength between the impurity and fermions in the Fermi sea. This interaction strength is given by the low-energy scattering length  $a_s$ . The latter is determined by the evaluation of the tree-level exchange of the molecule field  $\phi$  in the two-body problem where  $\mu_{\downarrow,\uparrow} = 0$ . This results in the initial value

$$m_{\phi\Lambda}^2 = -\frac{h^2}{8\pi a_s} - \frac{h^2}{2\pi^2} \int_0^{\infty} dp p^2 \left[ \frac{1}{2p^2 + R_{\uparrow}^{\Lambda} + R_{\downarrow}^{\Lambda}} - \frac{1}{2p^2} \right].$$

For large cutoff scales  $\Lambda$  this implies the scaling  $m_{\phi\Lambda}^2 \sim \mu(r)h^2\Lambda$  with  $\mu(r)$  being a regulator dependent number. It is the non-relativistic equivalent to the UV scaling discussed earlier, cf. Eq. (9). Furthermore, while the initial value of  $S_{\downarrow}$  is determined by its classical value  $S_{\downarrow\Lambda} = 1$ , we choose  $S_{\phi} = 0$  so that the bosonic field becomes a pure auxiliary field.

The Fermi momentum allows to define the dimensionless interaction parameter  $1/(k_F a_s)$ . In the following we work in units where  $k_F = 1$ . Finally the initial value of  $m_{\downarrow\Lambda}^2$  has to be chosen such that the self-energy acquired by the impurity leads to the fulfillment of the infrared condition  $m_{\downarrow k=0}^2 = 0$ . This condition ensures that the system is just on the verge of occupying a finite number of impurity atoms, which is the defining property of the impurity problem. This condition implies  $E_p = -m_{\downarrow\Lambda}^2 = \mu_{\downarrow}$  [54].

## C. Regulator Dependencies

*Shape dependence.* First we study the dependence of the results on the shape of the regulators when integrating out the bosonic and fermionic fields synchronously for the choice  $c_{\phi} = c_{\downarrow} = c_{\uparrow} = 1$ . Specifically, we monitor the energy of the polaron,  $E_p$ . In Fig. 13 we show the result for the polaron energy at unitary interactions,  $k_F a = \infty$ , as a function of the shape parameters  $n \equiv n_{\uparrow} = n_{\downarrow} = n_{\phi}$  (blue line). We have studied such a variation previously in Section III C in the context of a relativistic  $\varphi^4$  theory. Here, we choose  $b_i = 0$  and  $a_i = 100$  so that the regulator interpolates between a masslike  $k^2$  and a sharp regulator. The black dashed line corresponds to the result obtained from diagrammatic Monte Carlo,  $E_p/\epsilon_F \approx -0.615$  [56]. We note that a non-selfconsistent T-matrix approximation yields the result  $E_p/\epsilon_F \approx -0.607$  [66]. This approximation (leading order  $1/N$  expansion) corresponds to the se-



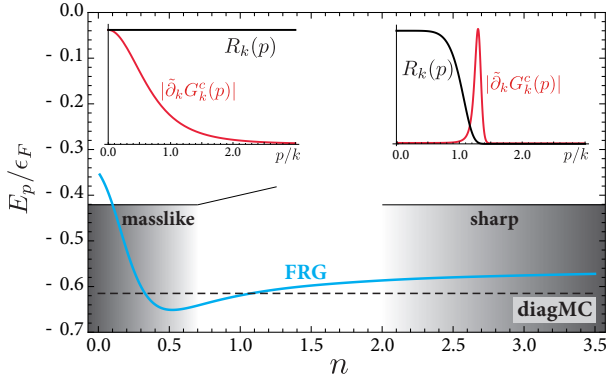


FIG. 13: Dependence of the polaron energy  $E_p/\epsilon_F$  on the regulator shape. The FRG results (blue) are shown for a crossover from a  $k^2$  to a sharp regulator by changing the exponent  $n \equiv n_i$  in all regulators at constant prefactors  $a_i = 100$  and  $b = 0$ . The exact result from diagrammatic Monte Carlo [56] is shown as dashed black line. The insets illustrate the structure of the regulator  $R_\downarrow(p)$  and single scale propagator  $\tilde{\partial}_k G_{\downarrow,k}^c = -(G_{\downarrow,k}^c)^2 \partial_k R_k$  in dependence of momentum  $p$ .

quence where first the dimer selfenergy is evaluated and then inserted into the self-energy of the impurity.

The result from the FRG calculation is shown as blue line. We observe a strong regulator shape dependence for small values of  $n$  (masslike regulator) as, here, the regulator leads to a non-local integration of field modes in momentum space.

In contrast, for  $n \rightarrow \infty$  (sharp regulator), the regulator becomes very local in momentum, and the results show only a small shape dependence. In the flows, the single scale propagator  $\tilde{\partial}_k G_k^c = -(G_k^c)^2 \partial_k R_k$  determines the locality of the regulator in momentum space which is illustrated in the insets in Fig. 13. Here, we show the form of the single scale propagator as well as  $R_\downarrow(p = |\mathbf{p}|)$  evaluated at zero frequency  $\omega = 0$ . We emphasise that momentum locality of the loop integration is but one of the important conditions for the optimisation. Regularity of the flow is a further important one, and the sharp cutoff fails in this respect. Indeed, it is the latter property which is crucial for critical exponents.

For  $n \rightarrow 0$  the non-local structure of field integration leads to a great sensitivity of the RG flow of  $\Gamma_k$  in theory space and hence a large sensitivity to the truncation chosen. This also finds an interpretation in terms of physics: due to the non-local structure of  $R_k(p)$  the RG flow does not separate between the few-body (vacuum) physics at large momenta on the one hand and on the other hand the emergence of correction to the vacuum flow due to finite density at small momenta. Such an unphysical mixture of physically vastly separated energy scales leads to an artificially strong dependence of the FRG results. We indicate the regime of artificial non-local, non-physical regulator choices by the grey shaded area in Fig. 13. The results illustrate the significance of the statement that for local truncations non-local cutoffs are a particularly

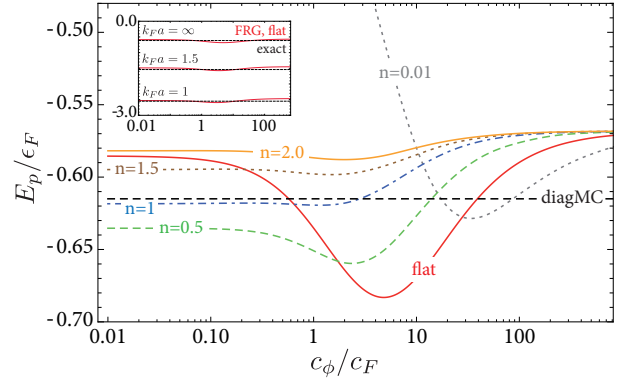


FIG. 14: Dependence of the polaron energy  $E_p/\epsilon_F$  on the relative bosonic-fermionic cutoff scale for various choices of the regulator shape. Results are shown for parameters as employed in Fig. 13 and also for the flat (Litim) regulator, cf. Table I. The inset shows the relative cutoff scale dependence for the flat regulator for various interaction strengths  $1/k_F a \neq 0$ .

bad choice of regularisation of RG flows. Instead general RG flows should be kept sufficiently local. However, also an extremely local regulator such as the sharp regulator is not desired as it prevents an interference of closely related momentum/length scales; it lacks regularity. We have indicated this regime as shaded area for  $n > 2$  where the single scale propagator becomes strictly peaked and interference of close-by momentum scales is heavily suppressed. The non-shaded regime corresponds to regulator choices which satisfy the criterion of sufficient interference of momentum scales while still avoiding an unphysical, non-local flow.

In summary, both extrem choices lack crucial properties of optimised FRG flows. This is also reflected in the fact that both limits do not do well in the optimisation criterium in its representations (32) and (37). Indeed, the combination of a sharp cutoff and a mass cutoff gives the worst result within the optimisation as it combines both failures, momentum nonlocality and lack of regularity.

*Dependence on relative cutoff scale.* Next, we investigate the dependence of the results on the relative scale at which the fermionic and bosonic degrees of freedom are integrated out. As in Section IV, this is achieved by changing the parameter  $c_\phi$  relative to the choice of  $c_F \equiv c_\downarrow = c_\uparrow$  in Eq. (53). The result is shown in Fig. 14 where we choose the same regulator shape parameters as in Fig. 13. Also we show the result for a flat regulator choice (red curves).

For  $c_\phi/c_F \rightarrow \infty$  the flow is equivalent to a purely fermionic flow since here the auxiliary bosons are integrated out only in the last step of the RG flow. In this last step the fermions are not subject to an RG gap  $R_\sigma$  anymore. In consequence, since the flow of the boson propagator is solely dependent on fermions, its self-energy  $\Sigma_\phi$  reached its final RG value already before this last RG step

is taken. This leads to results which are independent of the regulator shape,  $E_p/\epsilon_F \rightarrow -0.57$ , and which is the result obtained from an leading order  $1/N$  expansion [67] within our truncation for the momentum dependence of the bosonic propagator  $P_\phi$ .

Contrary, for  $c_\phi/c_F \rightarrow 0$  the bosonic field is integrated out first. The flow of the boson propagator, being only a functional of the fermionic Green's functions, is then completely suppressed in the first stage of the RG flow. This stage correspondingly amounts to a mere reversion of the introduction of the bosonic field  $\phi$  as an auxiliary degree of freedom mediating the atom-atom interaction. Since  $P_\phi$  cannot acquire any momentum dependence in this step, the resulting – now purely fermionic – theory has a truncation with a completely momentum independent coupling constant  $\lambda \sim -h^2/m_{\phi,\Lambda}^2$ .

Such a truncation of the effective flowing action  $\Gamma_k$  is of course a very poor one so that strong regulator dependencies are expected as also observed in Fig. 14. This result also represents an example supporting the discussion given in Section II B, cf. also Fig. 2: by choosing a poor truncation the flow is particularly sensitive to its path in theory space and hence can lead to strong regulator dependences of infrared quantities. Furthermore this result reflects the observation that the integrability condition (12) is more severely violated when the effective action is truncated to a larger degree (here by losing the momentum dependence of interactions altogether).

Having shown that the regulator dependences in the two extreme limits for  $c_\phi/c_F$  can be understood in simple terms we now turn to the intermediate regime where the bosonic and fermionic fields are integrated out synchronously. In this regime we observe a variation of the result for  $E_p/\epsilon_F$  on the order of  $\pm 10\%$ , with the exact result  $E_p/\epsilon_F = -0.615$  being in the vicinity of the predicted result by the FRG.

We also show the result when applying the flat regulator (red line) which shows similar variations with the relative cutoff scale. Our results indicate that within the regime of an ‘informed regulator choice’, indicated by the non-shaded region in Fig. 13, regulator dependences in FRG flows might allow for determining an error estimate on its own predictions.

## VI. CONCLUSION

In this work, we have presented a systematic investigation of the impact of different regulator choices on renormalisation group flows in given approximation schemes. To this end, we studied the functional RG which is based on the scale-dependent effective action. As an important aspect, this exact flow equation clearly exhibits the role played by the regulator within the RG, see (2), as it is directly proportional to its scale derivative. This already indicates the need for a thorough understanding of regulator dependencies. Such an understanding is not only important to the functional RG, in particular, but, more

comprehensively, extends to the analysis of approximations schemes in the renormalisation group framework in general.

Here, we focused on three key aspects of how the regulator choice affects RG results: First, we discussed how the choice of a specific regulator influences FRG flows by integrating over flow trajectories along closed loops in the space of action functionals varying both, the regulator scale and its shape function. For these flows we have discussed an integrability condition, [7], which is violated in the presence of truncations. Consequently, an educated regulator choice is mandatory to extract the best possible results from the RG in a given truncation. To this end we have extended the work on functional renormalisation in [7]. For the construction of such an optimised regulator, we have introduced the definition of the length of a RG trajectory which is minimal for an optimised regulator. This provides a pragmatic optimisation procedure which at the example of a single scalar field yields the flat regulator as a unique and analytical solution. A comparison of the lengths of these trajectories can also be set up straightforwardly in more complex models in order to identify optimised regulators. We leave explicit applications of this procedure for future work.

As a second aspect, we have investigated systems with two field degrees of freedom which both have to be regularised. Here a choice of relative cutoff scales is required. In given momentum-independent approximations this choice has a severe impact on the RG results and, hence, for the described physics. At the example of relativistic bosonic two-field models, we have discussed the consequences of a variation of the relative cutoff scales as well as its truncation dependence. We have shown that a crossover between different universality classes can be induced, triggered by the regulator-dependence of physical parameters in truncated flows. This entails that the relative cutoff scale has to be chosen carefully for a reliable description of a physical system in a given approximation. A controlled approach toward devising an optimised choice of relative cutoff scales can be provided by our optimisation procedure.

Third, we also have exhibited corresponding dependencies on relative cutoff scales and regulator shapes in non-relativistic continuum models of coupled fermionic and bosonic fields. At the example of the Fermi polaron problem in three spatial dimensions, we have illustrated such dependences and showed how to interpret them in physical terms. We suggested that, in the regime of an informed regulator choice, regulator dependences in FRG flows can provide error estimates. This has been discussed here at the example of a coupled non-relativistic many-body model. It will be interesting to investigate these capabilities further in more elaborate many-body models. Finally, it is of great interest to extend the functional optimisation framework layed out here and in [7] to an approach for general systematic error estimates in the functional RG.

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## Appendix A: Effective Action and RG Transformations

Note also, that the above renormalisation scheme dependence carries over to the full quantum effective action  $\Gamma[\phi] = \Gamma_{k=0}[\phi]$ . It satisfies the standard homogenous RG equation

$$s \frac{d\Gamma_{k=0}[\phi]}{ds} = 0, \quad (\text{A1})$$

(A1) is non-trivially achieved as all correlation functions  $\Gamma^{(n)}$  transform according to the anomalous dimension of the fields,

$$(\partial_s + n\gamma_\phi) \Gamma^{(n)} = 0, \quad \text{with} \quad \gamma_\phi \phi = s \frac{d\phi}{ds}. \quad (\text{A2})$$

For the purpose of the present work the RG-transformations of the full effective action are not relevant. Hence, from now we shall identify observables that are identical up to RG transformation of the underlying theory. Note however, that this identification does not remove the relevant UV scaling carried by (9).

## Appendix B: General One-Parameter Flows

We have introduced one-parameter flows, referring to general changes of the cutoff scale  $k$  with  $k(s)$ , changes of the shape of the regulator,  $r_s$  as well as reparametrisations of the theory. The corresponding flow equation has the same form as that for the  $k$ -flow in (2). It reads

$$s \frac{d\Gamma[\phi, R]}{ds} = \frac{1}{2} \text{Tr} G[\phi, R] (\partial_s + 2\gamma_\phi) R, \quad (\text{B1})$$

where the total derivative w.r.t.  $s$  also includes reparametrisations of the fields with  $d\phi/ds = \gamma_\phi \phi$  reflected in the term proportional to the anomalous dimension  $\gamma_\phi$  on the right hand side of (B1). The representation of the total  $s$  derivative similar to (14) is simply given by

$$\frac{d}{ds} = \left( -\frac{1}{2} \text{Tr} G[\phi, R] (\partial_s + 2\gamma_\phi) R G[\phi, R] \frac{\delta^2}{\delta\phi^2} \right). \quad (\text{B2})$$

Note that (B2) has to vanish as an operator if it represents a reparameterisation of the theory at hand, that is a standard renormalisation group transformation in the presence of a regulator. We infer, [7],

$$(\partial_s + 2\gamma_\phi) R \stackrel{!}{=} 0. \quad (\text{B3})$$

Eq. (B3) entails that the regulator has to be transformed as a two-point correlation function under RG-transformations in order to fully reparameterise the theory.

## Appendix C: Integrability Condition and Self-Consistency of Approximations

In case the integrability condition (15) holds, the flow necessarily has a (local) representation as a total derivative w.r.t.  $s$ , and hence it can be written as a total derivative of a diagrammatic representation. This entails that the integrated flow has a diagrammatic representation in terms of full vertices and propagators in the given approximation to the effective action  $\Gamma_k$ .

A simple example for such an approximation is perturbation theory: at perturbative  $n$ -loop order the integrated flow simply reproduces renormalised perturbation theory within a generalised BPHZ-scheme. Note however that the ordering scheme is an expansion in the fundamental coupling of the theory for both, the effective action and the flow equation, rather than one in expansion coefficients of the effective action such as the vertex expansion in terms of  $\Gamma_k^{(n)}$ . A more interesting example are 2PI-resummation schemes such as 2PI perturbation theory or  $1/N$ -expansions: it has been worked out how to implement renormalised versions of these schemes in the FRG, see [32, 33]. Hence in this case integrated flows provide renormalised perturbative or  $1/N$  2PI-resummations, and the integrability condition (15) is satisfied to any order of such an expansion. Again we note that the ordering scheme is an expansion in the fundamental coupling of the theory or the number of fields for both, the effective action and the flow equation. Similarly it is possible to find approximation schemes that lead to renormalised solutions of Dyson-Schwinger equations.

We emphasise that in both cases discussed above the flow operators (14), (B2) evaluated on the solution of the effective action in the given approximation, does not satisfy the integrability condition. In the case of  $n$ -loop perturbation theory the flow operators (14), (B2) then generates  $n$ -loop FRG-resummed perturbation theory which fails to satisfy (15). In the case of the  $n$ -loop 2PI approximation, the flow operators then generate  $n$ -loop FRG-resummed 2PI perturbation theory. To summarise, the violation of the integrability condition is a measure for the incompleteness, in terms of the full quantum theory, of fully non-perturbative resummation schemes.

## Appendix D: Threshold Functions

### Scalar Model

The scalar model from Sec. IID requires the threshold function  $I(\omega)$  defined in (19). Here, we explicitly give the analytical expressions for this integral for the cases of the flat regulator  $r_L$  and the sharp regulator  $r_{\text{sharp}}$ . For the flat regulator, we obtain  $I(\omega) \rightarrow I^{(L)}(\omega)$

$$I^{(L)}(\omega) = v_d \frac{4}{d} \frac{1}{1+\omega}. \quad (\text{D1})$$

Choosing the sharp regulator yields  $I(\omega) \rightarrow I^{(\text{sharp})}(\omega)$

$$I^{(\text{sharp})}(\omega) = -2v_d \log(1+\omega). \quad (\text{D2})$$

### Two-Field-Model

For the two-field models from Sec. IV, we have introduced similar threshold functions reading

$$I_{R,\phi}(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) = v_d \int_0^\infty y^{\frac{d}{2}+1} dy \left( -2r'_\phi(y) \right) \times \frac{y(1+r_\chi(y)) + \omega_\chi}{(y(1+r_\phi(y)) + \omega_\phi)(y(1+r_\chi(y)) + \omega_\chi) - \omega_{\phi\chi}}, \quad (\text{D3})$$

$$I_{R,\chi}(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) = v_d \int_0^\infty y^{\frac{d}{2}+1} dy \left( -2r'_\chi(y) \right) \times \frac{y(1+r_\phi(y)) + \omega_\phi}{(y(1+r_\phi(y)) + \omega_\phi)(y(1+r_\chi(y)) + \omega_\chi) - \omega_{\phi\chi}}, \quad (\text{D4})$$

and

$$I_{G,i}(x) = v_d \int_0^\infty y^{\frac{d}{2}+1} dy \frac{-2r'_i(y)}{y(1+r_i(y)) + x}, \quad (\text{D5})$$

with  $i \in \{\phi, \chi\}$ .

### Two-Field Model & Separate Cutoff Scales

For the discussion of the two-field model in Sec. IV, we use the flat regulator functions with a relative cutoff scale, as given in Eq. (47). With these shape functions, we obtain the threshold functions for the Goldstone modes

$$I_{G,\phi}(x) = \frac{4v_d}{d(x+1)}, \quad I_{G,\chi}(x) = \frac{4v_d c^{d/2+1}}{d(x+c)}. \quad (\text{D6})$$

The threshold function including radial modes are given by the expressions

$$I_{R,\phi}(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) = 2v_d \theta(1-c) F_1(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) + \frac{4v_d(c+\omega_\chi)(c^{d/2}\theta(1-c) + \theta(c-1))}{d((\omega_\phi+1)(c+\omega_\chi) - \omega_{\phi\chi})}, \quad (\text{D7})$$

$$I_{R,\chi}(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) = -2c v_d \theta(c-1) F_2(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) + \frac{4v_d(\omega_\phi+1)(c^{\frac{d}{2}+1}\theta(1-c) + c\theta(c-1))}{d((\omega_\phi+1)(c+\omega_\chi) - \omega_{\phi\chi})}, \quad (\text{D8})$$

where we have introduced the two integral functions

$$F_1(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) = \int_c^1 dy \frac{y^{\frac{d}{2}-1}(\omega_\chi + y)}{(\omega_\phi+1)(\omega_\chi + y) - \omega_{\phi\chi}}, \quad (\text{D9})$$

$$F_2(\omega_\phi, \omega_\chi, \omega_{\phi\chi}) = \int_c^1 dy \frac{y^{\frac{d}{2}-1}(\omega_\phi + y)}{(c+\omega_\chi)(\omega_\phi + y) - \omega_{\phi\chi}}. \quad (\text{D10})$$

This completes our list of required threshold functions for the two-field model with shape functions defined on separate cutoff scales.

## Appendix E: Flow equations for the couplings

### Scalar Model

In the symmetric regime, where  $\kappa = 0$ , we obtain the flow of the coupling constants  $\partial_t \lambda_i = (\partial_t u)^{(i)}|_{\rho=0}$  from the  $i$ th derivative of  $\partial_t u(\rho)$  with respect to  $\rho$ .

Analogously, in the symmetry broken regime, where  $\lambda_1 = 0$  and  $\kappa > 0$ , we get

$$\partial_t \kappa = -\frac{(\partial_t u)'}{\lambda_2} \Big|_{\rho=\kappa}, \quad \partial_t \lambda_{i \geq 2} = (\partial_t u)^{(i)} + u^{(i+1)} \partial_t \kappa \Big|_{\rho=\kappa}.$$

### Two-Field Model

Projecting the flow equation on the definition of  $u$  gives us the system of beta functions for the couplings

$$\partial_t \kappa_\phi = -\frac{\lambda_{02}(\partial_t u)^{(1,0)} - \lambda_{11}(\partial_t u)^{(0,1)}}{\lambda_{20}\lambda_{02} - \lambda_{11}^2} \Big|_{\kappa_\phi, \kappa_\chi}, \quad \partial_t \kappa_\chi = -\frac{\lambda_{20}(\partial_t u)^{(0,1)} - \lambda_{11}(\partial_t u)^{(1,0)}}{\lambda_{20}\lambda_{02} - \lambda_{11}^2} \Big|_{\kappa_\phi, \kappa_\chi}, \quad (\text{E1})$$

and

$$\partial_t \lambda_{ij} = (\partial_t u)^{(i,j)} + u^{(i+1,j)} \partial_t \kappa_\phi + u^{(i,j+1)} \partial_t \kappa_\chi \Big|_{\kappa_\phi, \kappa_\chi},$$

where the field invariants  $\rho_j$  are understood to be evaluated at their scale dependent expansion points  $\kappa_j$ .

## Appendix F: Calculation of Explicit Loop Flows

### Explicit Loop flows I

Starting with Eqs. (21) and (23), we can calculate a flow which translates continuously from one regulator to another as long as both regulators are finite. These equations can be easily extended to an  $O(N)$  model

$$\begin{aligned} \frac{d}{ds}u &= J(u' + 2\rho u'') + (N-1)J(u') \\ &+ \frac{\partial_s k(s)}{k(s)}(-du + (d-2)\rho u'). \end{aligned} \quad (F1)$$

A commonly used regulator is the sharp regulator  $r_{\text{sharp}}(y) = c/y \theta(1-y)|_{c \rightarrow \infty}$ , which is infinite in  $[0,1]$ . In order to interpolate between  $r_{\text{sharp}}$  and other regulators in a continuous manner we need to extend our calculations. Here, we interpolate between  $r_{\text{sharp}}$  and  $r_L$  using an interpolation which shifts the cutoff scale in  $r_{\text{sharp}}$  by a factor  $a(s) \in [0,1]$ .

$$r_s(y) = r_L(y) + r_{\text{sharp}}(y/a(s)^2), \quad (F2)$$

Hence,  $a(s) \rightarrow 0$  causes  $r_{\text{sharp}}$  to vanish. On the other hand, if  $a(s) = 1$ , then  $r_{\text{sharp}}$  causes the regulator to diverge on  $[0,1]$  such that there is no residual influence of  $r_L$ . The threshold function can be decomposed into two parts

$$\begin{aligned} J(\omega) &= v_d \int_0^\infty y^{\frac{d}{2}} dy \frac{\frac{d}{ds}r_s(y)}{y(1+r_s(y)) + \omega} \\ &= J^A(\omega) + J^B(\omega), \end{aligned} \quad (F3)$$

where  $J^A$  contains the regulator derivative from  $r_{\text{flat}}$  such that it can be inferred from (D1)

$$\begin{aligned} J^A(\omega) &= \frac{k'(s)}{k(s)}(1-a(s)^d)I^{(L)}(\omega) \\ &= v_d \frac{k'(s)}{k(s)} \frac{4}{d} \frac{1}{1+w} (1-a(s)^d). \end{aligned} \quad (F4)$$

Similarly,  $J^B$  corresponds to the regulator derivative of  $r_{\text{sharp}}$  and can be calculated by inserting (D2)

$$\begin{aligned} J^B(\omega) &= \left( \frac{a'(s)}{a(s)} + \frac{k'(s)}{k(s)} \right) a(s)^{d-2} I^{(\text{sharp})}(\omega) \\ &= -2v_d \left( \frac{a'(s)}{a(s)} + \frac{k'(s)}{k(s)} \right) a(s)^{d-2} \log(1+\omega). \end{aligned} \quad (F5)$$

### Explicit Loop flows II

In case we insist on a linear superposition between  $r_{\text{sharp}}$  and other finite regulators, the solution of the flow equation will show a discontinuity. As soon as we allow for a small contribution from  $r_{\text{sharp}}$ , it will dominate over all

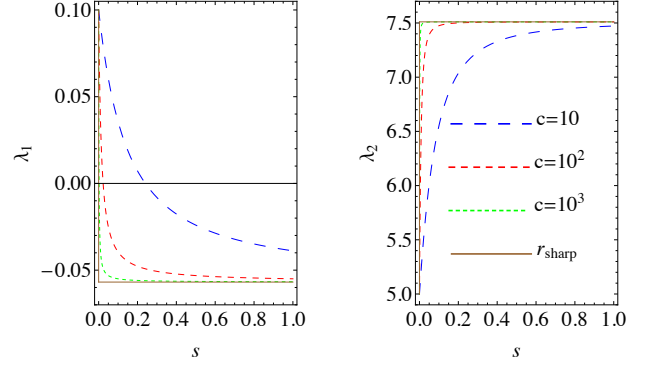


FIG. 15: Change of the initial conditions for different choices of interpolations between the flat regulator and the step like regulator, for various  $c$  and the sharp regulator.

finite regulators because it is infinite in the region  $[0,1]$ . This discontinuity can be seen in Figs. 6 and 15 for the couplings  $\lambda_1$  and  $\lambda_2$ . In order to calculate the magnitude of this discontinuity, we start at the flow equation at fixed  $k$  for an  $O(1)$ -model, Eq. (21)

$$\frac{d}{ds}u|_{k=\text{const}} = J(\omega), \quad \omega = u' + 2\rho u'', \quad (F6)$$

which we want to solve from  $s = 0$  to  $s = 1$ . Since the only change occurs at  $s = 0$ , we can simply denote the magnitude of the discontinuity  $\Delta u = u(s=1) - u(s=0)$ . The threshold function

$$J(\omega) = v_d \int_0^\infty y^{\frac{d}{2}} dy \frac{\partial_s r_s(y)}{y(1+r_s(y)) + \omega}, \quad (F7)$$

only depends on the change of shape  $\partial_s r_s(y)$ , but not on the scale change, since we are keeping  $k$  fixed. Inserting a linear superposition between  $r_{\text{sharp}}$  and  $r_L$

$$r_s(y) = (1-s)r_L(y) + s r_{\text{sharp}}(y), \quad (F8)$$

evaluates to

$$J(\omega) = v_d \int_0^1 y^{\frac{d}{2}-1} dy \frac{c(1+1/c-y/c)}{1+sc(1-1/c+y/c)+w} \Big|_{c \rightarrow \infty},$$

We now shift the flow variable from  $s \in [0,1]$  to  $\bar{s} = sc \in [0,c]$  and take the limit  $c \rightarrow \infty$ . The new modified flow equation, reading

$$\frac{d}{d\bar{s}}u|_{k=\text{const}} = \frac{2v_d}{d} \frac{1}{\bar{s}+1+w}, \quad (F9)$$

must be solved from  $\bar{s} = 0$  to  $\bar{s} = \infty$ . This equation leads to a logarithmic divergence if  $\bar{s} \rightarrow \infty$ . However, the divergent part is just a constant shift of the effective potential which can be removed by subtracting it

$$\frac{d}{d\bar{s}}u|_{k=\text{const}} = -\frac{2v_d}{d} \frac{1}{\bar{s}+1} \frac{\omega}{\bar{s}+1+\omega}. \quad (F10)$$

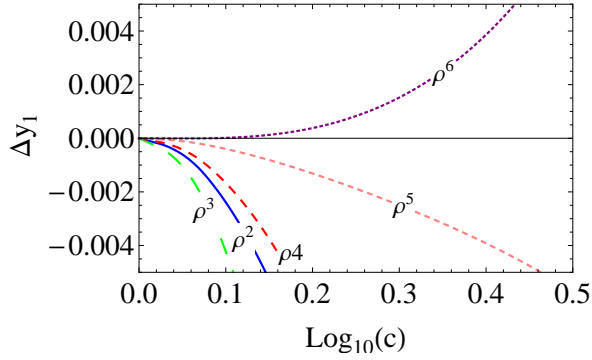


FIG. 16: Deviation from the critical exponent  $y_1$  using separated cutoff scales in the  $O(2) \oplus O(2)$  model in LPA from order  $\rho^2$  to order  $\rho^6$ .

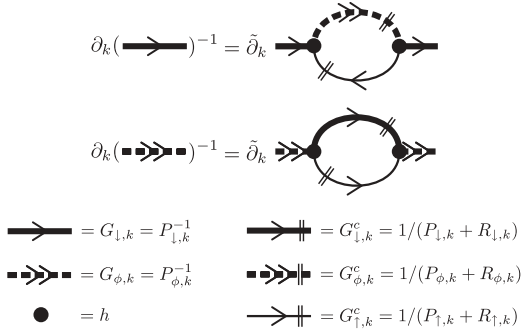


FIG. 17: Feynman diagrams for the flow of the dimer and impurity inverse propagators.

Our construction ensures that the discontinuity is expressed as  $\Delta u = u(\bar{s} = \infty) - u(\bar{s} = 0)$  which can be evaluated in a continuous flow equation.

## Appendix G: Two-Field-Model & Critical Exponents

In Fig. 16, we show a variant of Fig. 12, exhibiting the deviation of the critical exponent  $y_1$  in the  $O(2) \oplus O(2)$  model from its value at  $c = 1$  without taking the absolute value and the weighting factor from (48).

## Appendix H: Flows for the Polaron Problem

The flow equations, graphically represented in Fig. 17, are given by

$$\begin{aligned} \partial_k P_{\downarrow,k}(Q) &= h^2 \tilde{\partial}_k \int_P G_{\phi,k}^c(P) G_{\uparrow,k}^c(P+Q) \\ \partial_k P_{\phi,k}(Q) &= -h^2 \tilde{\partial}_k \int_P G_{\downarrow,k}^c(P) G_{\uparrow,k}^c(Q-P), \end{aligned} \quad (\text{H1})$$

where  $P \equiv (\omega, \mathbf{p})$ . The flowing inverse propagators  $P_k \equiv G_k^{-1}$  on the left-hand side are defined without the regulators, while the regulated propagators  $G_k^c$  are given by

$$G_k \equiv 1/P_k \quad G_k^c \equiv 1/(P_k + R_k), \quad (\text{H2})$$

and  $\tilde{\partial}_k$  implies that the derivative acts only on the regulator term  $R_k$  inside the cutoff propagators  $G_k^c$ .

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